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Association schemes on 28 points as mergings of a half-homogeneous coherent configuration

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Abstract

We consider a rank 112 coherent configuration $S = AP(2)$ on 28 points with 7 fibers of size 4. We describe S both axiomatically and as a model arising via the regular action of E_8 on the set of all 2-element subsets of an 8-element set. Moreover, we prove that our model is the unique structure, up to isomorphism, which satisfies the established axioms.

A most important feature of S is that its group $A\mathrm{Aut}(S)$ of algebraic automorphisms contains as a non-normal subgroup of index 8 the subgroup induced by all color automorphisms of S. This leads to a new type of automorphism of S, which we call "proper algebraic". All homogeneous mergings of S are described by us with the aid of a computer. Here, special attention is paid to so-called "algebraic mergings", i.e., those which arise from suitable subgroups of $A\mathrm{Aut}(S)$. As a result we are able to give a unified explanation of various association schemes on 28 points, including those of pseudocyclic and quasithin type, plus some of pseudotriangular type. Moreover, we provide computer-free proofs that these schemes are in fact attainable via appropriate mergings of classes from S.

Another interesting phenomenon is the existence of many "twins", i.e., pairs of non-isomorphic association schemes which are algebraically isomorphic inside S. Notable examples of twins are the triangular graph $T(8)$ paired with one of the Chang graphs, and the Mathon pseudocyclic scheme paired with the pseudocyclic scheme of Hollmann. In all, we decribe four pairs of twins and one set of triplets in rather great detail.

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1. Introduction

Coherent configurations and association schemes form a kernel of algebraic combinatorics, as it is defined in the seminal book [3] of Bannai and Ito. From the beginning, two important, and in a sense orthogonal, directions emerged: the derivation of infinite families of such objects of specific type (started in [3] and continued in [5]) and the creation of catalogues of all objects of a prescribed order (e.g., see [18]). Regarding the latter activity, an eventual goal is to explain all catalogued results in a way that is both natural and free of computer-dependent description. This is exactly our point of departure for this paper.

Specifically, we are interested in association schemes which are invariant with respect to a certain intransitive action of the elementary abelian group E_8. Namely, we begin with the regular action of E_8 on an eight-element set X, and consider its induced action on the set Ω of all two-element subsets of X. Clearly this is a degree 28 action, and the set S of 2-orbits of (E_8, Ω) is a coherent configuration of order 28 (with seven fibres of size 4) and rank 112. We are interested in all homogeneous coherent fusions of S, i.e., association schemes which arise via merging of classes of S. For reasons that will soon become apparent, we denote the above-mentioned coherent configuration by $AP(2)$.

In the interests of motivation, we now provide a brief historic account of some of the schemes we shall encounter in the course of our investigation. These include the pseudocyclic and quasithin association schemes on 28 points, plus some of pseudotriangular type.

Recall first the well-known family of triangular graphs $T(n)$ which (for $n \geq 4$) have the parameters $v = \binom{n}{2}$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$. A classical result asserts that a strongly regular graph with such parameters is unique up to isomorphism whenever $n \neq 8$ [8,23]. In the case of $n = 8$ there exist three additional graphs, commonly called the Chang graphs [6,7] (see also [38] for a nice interpretation of these graphs using the notion of Seidel switching).

In [31] Mathon outlined an infinite series of pseudocyclic association schemes coming from conics in $PG(3, q)$. When $q = 8$, the corresponding scheme has order 28 and three classes.

In [25] Hollmann introduced a certain kind of switching, which when applied to Mathon's scheme leads to a non-Schurian scheme of order 28 (see also [38]). This scheme is algebraically, but not combinatorially, isomorphic to the original scheme of Mathon which is Schurian. It turns out that Hollmann's scheme is the only one which shares the same tensor of structure constants as the scheme of Mathon. (A proof may be found in [24].)

Nearly twenty years later, M. Hirasaka and M. Muzychuk began their investigation of quasithin association schemes. In [22] they proved that for p a prime all quasithin association schemes of order $4p$ are Schurian with one possible exception: a certain set of parameters corresponding to $p = 7$. Consulting the computer catalogue of Hanaki and Miyamoto [18], they observed that there are just two association schemes with such parameters, one Schurian, the other non-Schurian. These schemes carry the respective designations #175 and #176 in [18].

Finally, in [2] elements of a new theory of extensions of association schemes are developed, one of the main motivations being to explain the occurrence of these association schemes of order 28.

In the present paper our aim is to describe all aforementioned association schemes on 28 points from a unique and unifying perspective. For this purpose we employ the recently formulated notion of "half-homogeneous coherent configuration", developed by Muzychuk in [35] for studying the two quasithin schemes on 28 points.[1] We focus on a specific class

[1] Note that as a precursor to [35], the Wallis–Fon-Der-Flaass prolific construction of strongly regular graphs uses a certain type of half-homogeneous coherent configuration, though in implicit form. (See [40,15].)

of such configurations, namely those of so-called "affine–projective type". The reason for this designation is that each such configuration may be constructed from a projective plane of order n and $n^2 + n + 1$ affine planes of order n, one plane per fibre. (Note that one may choose different planes for different fibres.) In general, this procedure leads to non-isomorphic configurations; however a unique one results when $n = 2$. Thus we are justified in denoting it by $AP(2)$.

In fact, our main focus here will be on $AP(2)$, an order 28 half-homogeneous coherent configuration which has a rich group of algebraic automorphisms. Many well-known association schemes on 28 points can be obtained via mergings of relations in $AP(2)$. Moreover, such schemes tend to occur in classes, where the members of each class are algebraically, but not combinatorially, isomorphic. Given any such class of size at least 2, we refer to any pair of its members as "twins". Thus, for each pair of twins one member is mapped onto the other via an algebraic automorphism of $AP(2)$ which is not induced by any combinatorial automorphism. We find this to be quite remarkable, considering the fact that many pairs of twins consist of one Schurian member and one non-Schurian member.

During the initial stages of our investigation, all results obtained depended heavily on the use of computers, in particular the packages GAP [37], GRAPE [39], nauty [32] and COCO [12]. Subsequently, we were able to supplant all computer-generated information by purely theoretical arguments involving, in particular, Galois correspondence (between permutation groups and relational structures), analysis of centralizer algebras, and various notions of "switching" within strongly regular graphs. We emphasize that our approach may be easily generalized to arbitrary elementary abelian groups, thus allowing for future investigations of this nature.

The balance of the paper is organized as follows. Section 2 contains all preliminary notions on which future sections depend. In Section 3 we introduce the notion of a half-homogeneous coherent configuration and that of a thin residue fission of an association scheme. Algebraic automorphisms of $AP(2)$ are discussed in Section 4, and homogeneous fusions of $AP(2)$ are treated in Section 5. In Section 6, we present five pairs of twins which provide nice illustrations of the general theory, and which, in our opinion, are of strong independent interest. Finally, in Section 7 we provide a quite wide context in which our current work may be viewed. In particular, we discuss the notion of a WFDF configuration (a certain type of half-homogeneous coherent configuration) and the indispensable role it plays in our present methodology. For now we simply mention that $AP(2)$ is an example of exactly this type of configuration.

As a final note, we draw attention to the existence of an Appendix A the purpose of which is to allow the reader to witness our computational data, and even replicate it if desired.

2. Preliminaries

We assume the reader's familiarity with the basic notions of coherent configuration [19, 41] and association scheme [3,5,44]. Although we have tried to make our presentation as self-contained as possible, we recommend [3,5] (see also [13]) as excellent sources of supplemental reading and information not explicitly provided within our text.

2.1. Zieschang notation and terminology

Among the differing notational styles which exist for coherent configurations and association schemes, we prefer in this paper to use the one introduced by Zieschang in [44]. In fact, this style was originally designed solely for association schemes; however, it is readily adaptable to coherent configurations.

Let (Ω, S) be a coherent configuration defined over the set Ω with corresponding set S of relations. We shall denote the structure constants of (Ω, S) by p_{rst} $(r, s, t \in S)$. For any $s \in S$, we define

$s^* := \{(\alpha, \beta) \mid (\beta, \alpha) \in s\}$;
$\omega s := \{\alpha \in \Omega \mid (\omega, \alpha) \in s\}$;
$D(s) := \{\omega \in \Omega \mid \omega s \neq \emptyset\}$;
$R(s) := D(s^*)$;
σ_s to be the adjacency matrix of $s \in S$.

When (Ω, S) is homogeneous (i.e., an association scheme), we shall replace σ_s with the more conventional notation $A(s)$ for the adjacency matrix of s.

Given any $r, s \in S$, we define their *relational product* to be $rs := \{t \in S \mid p_{rst} \neq 0\}$.

For any $\omega \in D(s)$, we define $n_s := |\omega s|$. We call a relation $s \in S$ *thin*[2] if $n_s = 1$.

We call a coherent configuration (Ω, S) *quasithin* provided $n_s \leq 2$ for every $s \in S$.

Given a subset $\Delta \subseteq \Omega$, we define i_Δ to be the diagonal relation on Δ.

A subset $\Delta \subseteq \Omega$ is called a *fibre* of S if $i_\Delta \in S$. We denote the set of fibres of S by $\mathcal{F}(S)$; thus $\bigcup_{\Delta \in \mathcal{F}(S)} i_\Delta = i_\Omega$. Note that (Ω, S) is homogeneous if and only if $|\mathcal{F}(S)| = 1$. For a fibre $\Delta \in \mathcal{F}(S)$ we set $S_\Delta := \{s \in S \mid R(s) = D(s) = \Delta\}$.

Given a field $\mathbb{F}$, the $\mathbb{F}$-linear span of the matrices σ_s, $s \in S$, is a subalgebra of the full matrix algebra $M_\Omega(\mathbb{F})$. We call it the *adjacency algebra* of S, and denote it by $\mathbb{F}[S]$.

Given coherent configurations (Ω, S), (Ω, S') over the common set Ω, we call S' a *fusion* of S (and S a *fission* of S') provided each relation $s' \in S'$ is a union of certain relations from S. In such a case, we often say that (Ω, S') arises via a *merging* of relations of (Ω, S). Equivalently, one could define S' to be a fusion of S provided $\mathbb{F}[S'] \subseteq \mathbb{F}[S]$. We express this fact notationally by writing $S' \sqsubseteq S$.

Given coherent configurations (Ω, S) and (Ω, T), we define $S \sqcap T$ to be the finest common fusion of S and T.

2.2. Association schemes

Let (Ω, S) be an association scheme. A nonempty subset $T \subseteq S$ will be called *closed* if $p^*q \subseteq T$ for any two elements $p, q \in T$. Given any closed subset T in S, we define $\omega T := \bigcup_{s \in T} \omega s$ for any $\omega \in \Omega$, and $s^T := \{(\alpha T, \beta T) \mid \beta \in \alpha s\}$ for any $s \in S$.

Given an association scheme (Ω, S) and closed subset $T \subseteq S$, we define

$$\Omega/T := \{\omega T \mid \omega \in \Omega\}, \qquad S/\!\!/T := \{s^T \mid s \in S\}.$$

In such a case $(\Omega/T, S/\!\!/T)$ is again an association scheme, called the *quotient scheme* of (Ω, S) with respect to T (see [44] for more details).

For $P, Q \subseteq S$ with $Q \neq \emptyset$, we define $K_P(Q) := \{p \in P \mid p^*Qp \subseteq Q\}$ to be the *strong normalizer of Q in P*. Given closed subsets U and T with $T \subseteq U \subseteq S$, we say that T is *strongly normal in U* if $U \subseteq K_S(T)$. The *thin residue* $\mathbf{O}^\theta(T)$ of T is now defined to be the intersection of all strongly normal closed subsets in T.

[2] Evdokimov and Ponomarenko refer to such a relation as *regular*.

2.3. 2-orbits of a permutation group

Given a permutation group (G, Ω), we denote by 2-orb(G, Ω) (or 2-orb(G), if G is specified as a subgroup of $\mathsf{Sym}(\Omega)$) the collection of all 2-orbits of (G, Ω), i.e., orbits of G in its induced action on the set $\Omega \times \Omega$. Clearly, the pair $(\Omega, \text{2-orb}(G, \Omega))$ forms a coherent configuration (moreover, an association scheme if the initial action (G, Ω) is transitive).

In many cases, it is difficult to obtain a reasonably transparent description of the set 2-orb(G, Ω). We mention in passing two important tools which, when used in conjunction, manage this task. The first is the computer package COCO (see [12,13]) which generates the desired information in terms of stored data; the second is the "method of subsequent splitting" (see Section 2.1.1.1 of [13]), a theoretical tool designed for the purpose of interpreting and explaining such data.

2.4. Isomorphisms and automorphisms

Let (Ω, S) and (Ω', S') be coherent configurations. A (*combinatorial*) *isomorphism* of (Ω, S) with (Ω', S') is a bijective mapping $\phi : \Omega \cup S \to \Omega' \cup S'$ for which $\Omega' = \Omega^\phi$, $S' = S^\phi$ and $(ws)^\phi = (w^\phi)s^\phi$ for all $s \in S$ and $w \in D(s)$.

It is a common practice in algebra to call an isomorphism from an object to itself an automorphism of the object. Unfortunately, this is not how the definition of automorphism of a coherent configuration arose historically. In our paper, we have chosen to follow tradition and never apply the term "automorphism" in this categorical sense.

We call an isomorphism $\phi : \Omega \cup S \to \Omega \cup S$ a (*combinatorial*) *automorphism* of (Ω, S) if ϕ acts as the identity mapping on S. An isomorphism of (Ω, S) with itself will be called a *colored automorphism* of (Ω, S). We denote the group of combinatorial automorphisms by $\mathrm{Aut}(S)$ and the group of colored ones by $C\mathrm{Aut}(S)$. Note that the elements of $C\mathrm{Aut}(S)$ are called *colored automorphisms* of S due to the fact that, unlike the elements of $\mathrm{Aut}(S)$, elements of $C\mathrm{Aut}(S)$ are allowed to permute colors (i.e., relations from S). As is well known, $\mathrm{Aut}(S)$ is a normal subgroup of $C\mathrm{Aut}(S)$, thus leading one to consider the natural action of the quotient group $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ on the set S.

An *algebraic isomorphism* between coherent configurations (Ω, S) and (Ω', S') is a bijection $\varphi : S \to S'$ which satisfies $p_{rst} = p_{r^\varphi s^\varphi t^\varphi}$. The $\mathbb{F}$-linear extension of an algebraic isomorphism yields an isomorphism between the adjacency algebras $\mathbb{F}[S]$ and $\mathbb{F}[S']$, which explains how this type of isomorphism gets its name. A well-known (folklore) result states that the restriction ϕ_S of a combinatorial isomorphism $\phi : \Omega \cup S \to \Omega' \cup S'$ yields an algebraic isomorphism between (Ω, S) and (Ω', S').

An algebraic isomorphism of (Ω, S) with itself is called an *algebraic automorphism* of (Ω, S) (alternatively, a *Bose–Mesner automorphism* in the terminology of Godsil; see [16]). We denote by $A\mathrm{Aut}(S)$ the group of all algebraic automorphisms of (Ω, S).

It is easy to verify that the group $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ embeds in $A\mathrm{Aut}(S)$. When this embedding is proper it leads to a distinction between combinatorially induced, and non-combinatorially induced, algebraic automorphisms. Crucial to our investigation is the occurrence of such a proper embedding in the case of the half-homogeneous coherent configuration $AP(2)$; see Section 4.3.

Since every algebraic automorphism permutes the subdiagonals (that is, relations of the form i_Δ, $\Delta \in \mathcal{F}(S)$), we obtain a natural action of $A\mathrm{Aut}(S)$ on the set of fibres defined via $i_{\Delta^g} := (i_\Delta)^g$, $g \in A\mathrm{Aut}(S)$.

2.5. Galois correspondence

Following in the spirit of Wielandt [43], with every coherent configuration (Ω, S) we may associate its automorphism group $\mathrm{Aut}(S) \le \mathsf{Sym}(\Omega)$, and with every permutation group (G, Ω) we may associate its coherent configuration of 2-orbits $(\Omega, \text{2-orb}(G))$. Such associations can be made formal by introducing the operators **Aut** and **2-orb**:

$$\mathbf{Aut} : (\Omega, S) \mapsto \mathrm{Aut}(S), \qquad \mathbf{2\text{-}orb} : (G, \Omega) \mapsto (\Omega, \text{2-orb}(G)).$$

Note that compositions of these two operators, namely,

$$\mathbf{2\text{-}orb} \circ \mathbf{Aut} : (\Omega, S) \mapsto (\Omega, \text{2-orb}(\mathrm{Aut}(S))), \quad \mathbf{Aut} \circ \mathbf{2\text{-}orb} : (G, \Omega) \mapsto \mathrm{Aut}(\text{2-orb}(G))$$

lead us back to the same set from which we started.

Proposition 2.1. *For a given fixed set Ω, the operators* **Aut** *and* **2-orb** *form a Galois correspondence (in the sense of* [1]*) in which the compositions* **2-orb** $\circ$ **Aut** *and* **Aut** $\circ$ **2-orb** *are the closure operators. In other words, given coherent algebras (Ω, S), (Ω, S') and permutation groups $G, G' \in \mathsf{Sym}(\Omega)$ the following hold:*

(i) $S \sqsubseteq S' \Longrightarrow \mathrm{Aut}(S') \le \mathrm{Aut}(S)$;
(ii) $G \le G' \Longrightarrow \text{2-orb}(G') \sqsubseteq \text{2-orb}(G)$;
(iii) $S \sqsubseteq \text{2-orb}(\mathrm{Aut}(S))$;
(iv) $G \le \mathrm{Aut}(\text{2-orb}(G))$.

An outline of a proof of Proposition 2.1 may be found in [29].

By definition, the Galois-closed objects are those coherent configurations (resp., permutation groups) which attain equality in (iii) (resp., (iv)). Galois-closed coherent configurations are called *Schurian*, while Galois-closed permutation groups are called *2-closed*.

2.6. Algebraic and Galois fusions

Let (Ω, S) be a coherent configuration. For each group G of algebraic automorphisms of (Ω, S) one can define an algebraic fusion of S in the following way. Let S/G denote the set of orbits of G on S. For each $O \in S/G$ define O^+ to be the union of all relations from O. Then the set of relations $\{O^+ \mid O \in S/G\}$ forms a coherent configuration on Ω. We call it the *algebraic fusion of S with respect to G*. This fusion is homogeneous if and only if G acts transitively on $\mathcal{F}(S)$. Note that if $G \le C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ and S is Schurian, then the resulting fusion is Schurian as well. In contrast, a fusion with respect to a subgroup *not* contained in $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ may well lead to a non-Schurian association scheme.

The Galois correspondence described above produces another type of fusion. Again, let (Ω, S) be a coherent configuration, and set $T := \text{2-orb}(\mathrm{Aut}(S))$. Then $S \sqsubseteq T$. Consider an arbitrary overgroup G of $\mathrm{Aut}(S)$ in $\mathsf{Sym}(\Omega)$, that is $\mathrm{Aut}(S) \le G \le \mathsf{Sym}(\Omega)$. Then $\text{2-orb}(G) \sqsubseteq T$. The coherent configuration $S \sqcap \text{2-orb}(G)$ is a fusion of S, which we shall call a *Galois fusion* of S. If S is Schurian, then $T = S$ and the fusion $S \sqcap \text{2-orb}(G)$ coincides with $\text{2-orb}(G)$ which is Schurian as well.

Note that in certain instances Galois and algebraic fusions produce the same configurations.

3. Half-homogeneous configurations

A non-homogeneous coherent configuration (Ω, S) is called *half-homogeneous* if all its fibres have the same cardinality. This terminology is derived from the fact that such structures appear naturally as coherent configurations of 2-orbits of half-transitive permutation groups.[3]

One source of half-homogeneous coherent configurations appears, though implicitly, in the Wallis–Fon-Der-Flaass prolific construction of strongly regular graphs; see [40,15]. A detailed analysis of their construction shows that many of the graphs obtained "live" inside certain such configurations.

A second source of half-homogeneous coherent configurations is a thin residue of an association scheme; see [35]. In the next subsection we give examples of such configurations on 28 points; in fact each one is isomorphic to $AP(2)$.

Last but not least, a third source of half-homogeneous coherent configurations occurs in [11], where Evdokimov and Ponomarenko study a certain class of such configurations with fibre size 4 from which they construct an infinite series of highly closed coherent configurations.

3.1. Two models of AP(2)

Example 1. Here, we construct $AP(2)$ starting from a half-transitive permutation group. Consider the elementary abelian group $E_8 \cong \mathbb{Z}_2^3$, and let Ω denote the set of all two-element subsets of E_8. Then E_8 acts on Ω in a natural way via $\{x, y\} \mapsto \{x + a, y + a\}$. Define an equivalence relation $\sim$ on Ω as follows: $\{x, y\} \sim \{u, v\} \iff x + y = u + v$. This relation partitions Ω into equivalence classes which are parameterized by the nonzero elements of E_8: the class Ω_v corresponding to the element $v \in E_8 \setminus \{0\}$ has the form $\Omega_v = \{\{x, y\} \mid x + y = v\}$. It is easy to see that Ω_v consists of all cosets of the subgroup $\langle v\rangle$; thus Ω_v is merely the quotient group $E_8/\langle v\rangle$. Note that each element of Ω may be identified with a coset of a subgroup of order two, and hence the sets Ω_v ($v \in E_8 \setminus \{0\}$) are just the orbits of the permutation group (E_8, Ω).

In order to describe the 2-orbits of (E_8, Ω) we introduce the auxiliary subsets $\Omega_{u,v} := \{\langle u\rangle, \langle u\rangle + v\} \subseteq \Omega_u$, where u, v are distinct nonzero elements of E_8. Note that

$$\Omega_{u,v} = \Omega_{u,u+v}.$$

The 2-orbits of (E_8, Ω) may now be described as the following relations:

$$\begin{array}{lcll}
\text{28 relations of the form:} & \widehat{\alpha} & := & \{(\omega, \omega + \alpha) \mid \omega \in \Omega_v\}, \quad \alpha \in \Omega_v,\ v \in E_8 \setminus \{0\} \\
\text{42 relations of the form:} & b^0_{uv} & := & (\Omega_{u,v} \times \Omega_{v,u}) \cup (\overline{\Omega}_{u,v} \times \overline{\Omega}_{v,u}), \quad u \neq v \in E_8 \setminus \{0\} \\
\text{42 relations of the form:} & b^1_{uv} & := & (\Omega_{u,v} \times \overline{\Omega}_{v,u}) \cup (\overline{\Omega}_{u,v} \times \Omega_{v,u}), \quad u \neq v \in E_8 \setminus \{0\},
\end{array}$$

where $\overline{\Omega}_{u,v}$ denotes the relative complement $\Omega_u \setminus \Omega_{u,v}$ of $\Omega_{u,v}$ in Ω_u. It is easy to see that

$$(\widehat{\alpha})^* = \widehat{\alpha}, \qquad (b^i_{uv})^* = b^i_{vu}.$$

Thus, we obtain a total of 112 relations of which exactly 28 are symmetric. Note also that this construction admits an evident generalization when E_8 is replaced be an arbitrary elementary abelian group. $\square$

[3] Following Wielandt [42], we call an intransitive permutation group (G, Ω) *half-transitive* if all its orbits have the same length. It is easy to see that this condition implies that all fibres of $(\Omega, \text{2-orb}(G, \Omega))$ have the same cardinality.

Example 2. Now we shall construct $AP(2)$ from a more axiomatic perspective. Define $\Delta :=$ $\{a, b, c, d\} \times \{1, 2, 3, 4, 5, 6, 7\}$.

The seven subsets $\Delta_j := \{a, b, c, d\} \times \{j\}$ will be the fibres of the coherent configuration we are going to build. For ease of notation, let us agree to abbreviate an ordered pair $(x, j) \in \Delta_j$ simply by x_j.

On each fibre Δ_j, we define four relations s_{ij}, $i = 0, 1, 2, 3$, as follows:

$$s_{0j} = \{(a_j, a_j), (b_j, b_j), (c_j, c_j), (d_j, d_j)\}, \quad s_{1j} = \{(a_j, b_j), (b_j, a_j), (c_j, d_j), (d_j, c_j)\},$$
$$s_{2j} = \{(a_j, d_j), (d_j, a_j), (b_j, c_j), (c_j, b_j)\}, \quad s_{3j} = \{(a_j, c_j), (c_j, a_j), (b_j, d_j), (d_j, b_j)\}.$$

Note that each s_{ij} induces a permutation on Δ_j, and $\{s_{0j}, s_{1j}, s_{2j}, s_{3j}\}$ is a regular permutation group on Δ_j isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In this manner we obtain 28 symmetric relations s_{ij}, where $0 \le i \le 3$, $1 \le j \le 7$.

Now consider the Fano plane $\mathcal{F}$ with point set $P = \{1, 2, 3, 4, 5, 6, 7\}$. As usual, lines of $\mathcal{F}$ may be specified as certain three-element subsets of P. Let us temporarily "color" the seven lines of $\mathcal{F}$ with $l_1, l_2, l_3, l_4, l_5, l_6, l_7$.

With each fibre Δ_j we may associate the three lines of $\mathcal{F}$ incident to point j. This allows us to define an arbitrary bijection between said lines and the relations s_{1j}, s_{2j}, s_{3j}. In this manner, each of our 21 nonreflexive relations receives a certain color l_t.

We now wish to define relations across different fibres.

Without loss of generality, let us assume $l_1 = \{1, 2, 3\}$. This means that each of the fibres Δ_1, Δ_2, Δ_3 admits a unique (nonreflexive) relation which has color l_1. Again without loss, assume the relations so arising in the first two fibres are s_{11} and s_{12}.

Our next task is to define relations across fibres Δ_1 and Δ_2 which are naturally induced from the relations s_{11} and s_{12}. We obtain four such relations s^i_{12}, s^i_{21}, where $i = 0, 1$:

$$s^0_{12} = (\{a_1, b_1\} \times \{c_2, d_2\}) \cup (\{c_1, d_1\} \times \{a_2, b_2\}),$$
$$s^1_{12} = (\{a_1, b_1\} \times \{a_2, b_2\}) \cup (\{c_1, d_1\} \times \{c_2, d_2\}),$$
$$s^0_{21} = (\{c_2, d_2\} \times \{a_1, b_1\}) \cup (\{a_2, b_2\} \times \{c_1, d_1\}) = (s^0_{12})^*,$$
$$s^1_{21} = (\{a_2, b_2\} \times \{a_1, b_1\}) \cup (\{c_2, d_2\} \times \{c_1, d_1\}) = (s^1_{21})^*.$$

Clearly, in this manner we obtain $7 \cdot \binom{3}{2} \cdot 4 = 84$ non-symmetric cross-fibre relations. Thus we get a total of 112 relations of which 28 are symmetric. $\square$

Remark. (i) Our construction in Example 2 is well defined because the Fano plane is a Steiner system $S(2, 3, 7)$; hence each pair of distinct fibres appears in the same line exactly once.

(ii) Evdokimov and Ponomarenko perform a similar construction in [11], where instead of the Fano plane they use an arbitrary cubic graph.

(iii) As the reader will observe, there is a great deal of freedom in our construction in Example 2. Nonetheless, it turns out that given any variance in the selection process, the resulting object is always a half-homogeneous coherent configuration isomorphic to $AP(2)$.

Regarding Remark (iii), we conclude this subsection with a simple argument which shows why any pair of coherent configurations as described in Example 2 must be isomorphic: The symmetric group S^j_4 defined over $\{a_j, b_j, c_j, d_j\}$ acts transitively on the set of six possible orderings of s_{1j}, s_{2j}, s_{3j}. Thus any two bijections between these relations and the three lines of $\mathcal{F}$ incident to j are conjugate by an element of S^j_4.

3.2. Thin residue fissions

Let (Ω, S) be an association scheme, and let $T \subseteq S$ be a closed subset. Following [10] we denote by $S_{[T]}$ the WL-closure of the following set of relations: $S \cup \{i_\Delta\}_{\Delta \in \Omega/T}$.

Let N be a strongly normal closed subset of S, that is $s^* N s \subseteq N$ for each $s \in S$. Note that we always have that $\mathbf{O}^\theta(S) \subseteq N$. It follows from the properties of a strongly normal closed subset that $\Delta s := \{\delta s \mid \delta \in \Delta\} \in \Omega/N$ for each $s \in S$ and $\Delta \in \Omega/N$. Thus every $s \in S$ induces a permutation $\widetilde{s} \in \mathsf{Sym}(\Omega/N)$: $\Delta^{\widetilde{s}} := \Delta s$. For each $s \in S$ and $\Delta \in \Omega/N$ we set $s_\Delta := s \cap (\Delta \times \Delta s)$. Note that $D(s_\Delta) = \Delta$ and $R(s_\Delta) = \Delta s$.

Proposition 3.1 (*[35]*). *For each $r, s \in S$ and $\Sigma, \Delta \in \Omega/N$ it follows that*

$$A(r_\Sigma)A(s_\Delta) = \begin{cases} 0, & \Sigma r \neq \Delta; \\ \sum_t p_{rst} A(t_\Sigma), & \Sigma r = \Delta. \end{cases} \tag{1}$$

In particular, $S_{[N]} =: \{s_\Delta \mid s \in S, \Delta \in \Omega/N\}$ is a half-homogeneous coherent configuration with fibre partition Ω/N. We call $S_{[\mathbf{O}^\theta(S)]}$ the thin residue fission of S.

Proof. If $\Sigma r \neq \Delta$, then $R(r_\Sigma) \cap D(s_\Delta) = \Sigma r \cap \Delta = \emptyset$ implying $A(r_\Sigma)A(s_\Delta) = 0$ as desired.

Assume now that $\Sigma r = \Delta$. Pick an arbitrary $(\alpha, \beta) \in t_\Sigma$ and consider two subsets

$$\Gamma := \{\gamma \in \Omega \mid (\alpha, \gamma) \in r_\Sigma \text{ and } (\gamma, \beta) \in s_\Delta\};$$
$$\Gamma' := \{\gamma \in \Omega \mid (\alpha, \gamma) \in r \text{ and } (\gamma, \beta) \in s\}.$$

Clearly $\Gamma \subseteq \Gamma'$ and $|\Gamma'| = p_{rst}$.

Thus, to finish the proof it suffices to show that $\Gamma' \subseteq \Gamma$. First observe that

$$\gamma \in \Gamma' \Rightarrow (\alpha, \gamma) \in r \Rightarrow \gamma \in \alpha r.$$

Since $(\alpha, \beta) \in t_\Sigma$, we have $\alpha \in \Sigma$ which implies

$$\gamma \in \alpha r \Rightarrow \gamma \in \Sigma r \Rightarrow (\alpha, \gamma) \in \Sigma \times \Sigma r \Rightarrow (\alpha, \gamma) \in r_\Sigma.$$

Similarly, $(\gamma, \beta) \in s \Rightarrow \beta \in \gamma s \subseteq \Sigma r s = \Delta s \Rightarrow (\gamma, \beta) \in s_\Delta$. Thus $\gamma \in \Gamma$ follows by definition. $\square$

The group $\widetilde{S}$ is a regular subgroup of $\mathsf{Sym}(\Omega/N)$. Therefore its centralizer in $\mathsf{Sym}(\Omega/N)$ is a regular subgroup of $\mathsf{Sym}(\Omega/N)$ isomorphic to $\widetilde{S}$. We denote it by G for brevity. Define the action of G on $S_{[N]}$ by $(s_\Delta)^g := s_{(\Delta^g)}$.

Proposition 3.2 (*[35]*). *The mapping $s_\Delta \mapsto (s_\Delta)^g$ is an algebraic automorphism of $S_{[N]}$.*

Proof. According to (1), $p_{r_\Delta s_\Sigma t_\Theta} = p_{rst}$ whenever $\Delta = \Sigma r$ and $\Theta = \Sigma$, and vanishes otherwise. Now the claim follows from the fact that G and $\widetilde{S}$ commute elementwise. $\square$

Let us now apply the above theory to the quasithin schemes #175 and #176 to show that in each case its thin residue fission is isomorphic to $AP(2)$. (In fact, we do this only for scheme #175 as a verification for #176 is similar.) To establish notation, we denote the underlying set of scheme #175 by $\Omega = \{1, \ldots, 28\}$, and its basis relations by $s_0, \ldots, s_{15}$. According to [18] the generalized adjacency matrix of #175 takes the following form:

0	1	2	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12	12	13	13	14	14	15	15
1	0	3	2	4	4	6	6	5	5	7	7	8	8	10	10	9	9	11	11	14	14	15	15	12	12	13	13
2	3	0	1	7	7	5	5	6	6	4	4	11	11	10	10	9	9	8	8	12	12	15	15	14	14	13	13
3	2	1	0	7	7	6	6	5	5	4	4	11	11	9	9	10	10	8	8	14	14	13	13	12	12	15	15
5	5	6	6	0	2	8	11	8	11	1	3	12	12	4	7	4	7	14	14	13	15	9	10	13	15	9	10
5	5	6	6	2	0	11	8	11	8	3	1	12	12	7	4	7	4	14	14	15	13	10	9	15	13	10	9
4	7	4	7	9	10	0	1	3	2	9	10	5	6	13	15	13	15	5	6	8	8	12	14	11	11	14	12
4	7	4	7	10	9	1	0	2	3	10	9	6	5	15	13	15	13	6	5	8	8	14	12	11	11	12	14
7	4	7	4	9	10	3	2	0	1	9	10	6	5	13	15	13	15	6	5	11	11	14	12	8	8	12	14
7	4	7	4	10	9	2	3	1	0	10	9	5	6	15	13	15	13	5	6	11	11	12	14	8	8	14	12
6	6	5	5	1	3	8	11	8	11	0	2	14	14	4	7	4	7	12	12	15	13	10	9	15	13	10	9
6	6	5	5	3	1	11	8	11	8	2	0	14	14	7	4	7	4	12	12	13	15	9	10	13	15	9	10
9	9	10	10	13	13	4	7	7	4	15	15	0	3	12	14	14	12	1	2	5	6	8	11	6	5	11	8
9	9	10	10	13	13	7	4	4	7	15	15	3	0	14	12	12	14	2	1	6	5	11	8	5	6	8	11
8	11	11	8	5	6	12	14	12	14	5	6	13	15	0	1	2	3	15	13	9	10	4	4	10	9	7	7
8	11	11	8	6	5	14	12	14	12	6	5	15	13	1	0	3	2	13	15	10	9	4	4	9	10	7	7
11	8	8	11	5	6	12	14	12	14	5	6	15	13	2	3	0	1	13	15	10	9	7	7	9	10	4	4
11	8	8	11	6	5	14	12	14	12	6	5	13	15	3	2	1	0	15	13	9	10	7	7	10	9	4	4
10	10	9	9	15	15	4	7	7	4	13	13	1	2	14	12	12	14	0	3	6	5	8	11	5	6	11	8
10	10	9	9	15	15	7	4	4	7	13	13	2	1	12	14	14	12	3	0	5	6	11	8	6	5	8	11
13	15	13	15	12	14	9	9	10	10	14	12	4	7	8	11	11	8	7	4	0	3	5	6	2	1	5	6
13	15	13	15	14	12	9	9	10	10	12	14	7	4	11	8	8	11	4	7	3	0	6	5	1	2	6	5
12	14	14	12	8	11	13	15	15	13	11	8	9	10	5	5	6	6	9	10	4	7	0	2	4	7	1	3
12	14	14	12	11	8	15	13	13	15	8	11	10	9	5	5	6	6	10	9	7	4	2	0	7	4	3	1
15	13	15	13	12	14	10	10	9	9	14	12	7	4	11	8	8	11	4	7	2	1	5	6	0	3	5	6
15	13	15	13	14	12	10	10	9	9	12	14	4	7	8	11	11	8	7	4	1	2	6	5	3	0	6	5
14	12	12	14	8	11	15	13	13	15	11	8	10	9	6	6	5	5	10	9	4	7	1	3	4	7	0	2
14	12	12	14	11	8	13	15	15	13	8	11	9	10	6	6	5	5	9	10	7	4	3	1	7	4	2	0

The thin residue $N := \mathbf{O}^\theta(S)$ of scheme #175 is comprised of four relations g_0, g_1, g_2, g_3 which form a group with respect to the usual relational product. (In fact, this is the famous Klein-four group $E_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.) Hence the classes of the partition Ω/N are merely the orbits of N. A direct computation yields $\Omega/N = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7\}$, where

$$\Omega_1 = \{1, 2, 3, 4\}, \quad \Omega_2 = \{5, 6, 11, 12\}, \quad \Omega_3 = \{7, 8, 9, 10\}, \quad \Omega_4 = \{13, 14, 19, 20\},$$
$$\Omega_5 = \{15, 16, 17, 18\}, \quad \Omega_6 = \{21, 22, 25, 26\}, \quad \Omega_7 = \{23, 24, 27, 28\}.$$

In the thin residue fission $S_{[N]}$, each of the relations s_i splits into a disjoint union of seven relations $(s_i)_{\Omega_j}$, $j = 1, \ldots, 7$. Thus $S_{[N]}$ contains $7 \cdot 16 = 112$ relations in all. For example, if we take s_8 and Ω_1 then $\Omega_1 s_8 = \{13, 14, 19, 20\} = \Omega_4$, and hence

$$\begin{aligned}(s_8)_{\Omega_1} = s_8 \cap (\Omega_1 \times \Omega_4) = \{&(1, 13), (1, 14), (2, 13), (2, 14), (3, 19),\\ &(3, 20), (4, 19), (4, 20)\}.\end{aligned}$$

In other words, $(s_8)_{\Omega_1}$ is a directed bipartite graph on $\Omega_1 \cup \Omega_4$ which, neglecting orientation of arcs, is isomorphic to $2 \circ K_{2,2}$ (two copies of the complete bipartite graph $K_{2,2}$). It turns out that this example is representative of the general case! That is, each of the seven parts of each s_i $(i = 0, 1, 2, 3)$ is an oriented $2 \circ K_{2,2}$. Thus $S_{[N]}$ is seen to have the same local structure as $AP(2)$. We may regard this as a first step in fulfilling our most immediate goal: to show that $S_{[N]}$ and $AP(2)$ are indeed isomorphic.

Proposition 3.3. *Let $S_{[N]}$, as above, denote the thin residue fission of scheme* #175. *To establish the desired isomorphism of $S_{[N]}$ with $AP(2)$, it suffices to show that the automorphism group of*

#175 contains an elementary abelian subgroup E of order 8 which has orbits $\Omega_1, \ldots, \Omega_7$ and satisfies the condition $E_\omega = E_{\omega'} \iff \omega' \in \omega^E$. (Here E_α denotes the stabilizer in E of $\alpha \in \Omega$, and α^E denotes the E-orbit containing α.)

Proof. As $\Omega_1, \ldots, \Omega_7$ are the orbits of E, the relations $i_{\Omega_1}, \ldots, i_{\Omega_7}$ are clearly E-invariant. But scheme S is E-invariant too. Thus $S_{[N]}$ is E-invariant, as it is the WL-closure of two sets of E-invariant relations.

Pick an arbitrary transversal $\omega_i \in \Omega_i$, $1 \le i \le 7$. Then $E_i := E_{\omega_i}$ are seven pairwise distinct subgroups of E of order 2. Therefore the action of E on Ω is isomorphic to the action of E on the set of the cosets $E/E_1 \cup \cdots \cup E/E_7$. But this is exactly the action of E_8 described in Section 3.1. Thus the configuration 2-orb(E) is isomorphic to $AP(2)$, implying that $S_{[N]}$ is a fusion of 2-orb(E). Now it follows from $|S_{[N]}| = 112 = |\text{2-orb}(E)|$ that $S_{[N]} = \text{2-orb}(E) \cong AP(2)$. □

The easiest way to find such a subgroup is to apply a computer program. However, by appealing to rudimentary arguments from the theory of quasithin association schemes, we can fashion a computer-free proof. More precisely, we shall derive the required subgroup from the generalized adjacency matrix of our scheme.

We need to first recall some basic facts about quasithin association schemes. Let (Ω, S) be a quasithin association scheme, that is, $n_s \in \{1, 2\}$ for each $s \in S$. Pick an arbitrary point $\omega \in \Omega$. Relative to ω, we may partition Ω into cells: ωs, $s \in S$. Since S is quasithin, each cell has either one or two elements. Let us collect the two-element cells of this partition, say $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$, and define an involutory permutation p_ω of Ω (see [21]) to be the product of transpositions: $p_\omega =: (x_1, y_1)(x_2, y_2) \cdots (x_n, y_n)$.

Considering now the generalized adjacency matrix A of an ambient quasithin scheme, we may interpret that an involutory permutation p_i corresponding to the ith row of A permutes j and k if and only if $A_{ij} = A_{ik}$, while leaving all other points fixed. For example, from the generalized adjacency matrix for the scheme #175 we obtain the permutation

$$\begin{aligned} p_7 = {} & (1,3)(2,4)(5,11)(6,12)(13,19)(14,20)(15,16) \\ & (16,18)(21,22)(23,28)(24,27)(25,26). \end{aligned}$$

By Lemma 3.5 of [21], we are assured that p_ω is an automorphism of the considered quasithin scheme. In particular, p_7 is an automorphism of scheme #175.

Among the 28 involutions obtained in this manner, only seven are distinct. More explicitly, $p_i = p_j$ if and only if i and j belong to the same orbit of N. Moreover, these seven involutions, together with the identity permutation, form a group which is elementary abelian as required by Proposition 3.3. Thus $AP(2)$ is a thin residue fission of #175. As previously mentioned, the same result holds for the quasithin scheme #176; in fact, if we repeat our procedure using the adjacency matrix of #176, we obtain that its thin residue fission is not only isomorphic to that of #175 but actually coincides with it! Thus we have the following.

Corollary 3.4. *Both #175 and #176 are fusion schemes of $AP(2)$.*

In Section 6.1, we shall see that each of #175 and #176 is an algebraic fusion with respect to a certain Sylow 7-subgroup of $A\text{Aut}(S)$; however, these Sylow 7-subgroups lie in different orbits of $C\text{Aut}(S)/\text{Aut}(S)$.

4. Automorphism groups related to $AP(2)$

Throughout this section, we denote by S the set of relations of the coherent configuration $AP(2)$, that is $S := 2\text{-orb}(E_8, \Omega)$, where Ω denotes the set of all two-element subsets of E_8.

4.1. CAut(S) and CAut(S)/Aut(S)

Our first result concerns the action (E_8, Ω) specified in Example 1. Recall that $\Omega = \cup_{v \in E_8 \setminus \{0\}} \Omega_v$ where $\Omega_v = E_8/\langle v \rangle$.

Proposition 4.1. *The action (E_8, Ω) is faithful and 2-closed.*

Proof. Denote by G the 2-closure of (E_8, Ω), and let u, v be distinct nonzero elements of E_8. Since $(\langle u \rangle, \langle v \rangle)^{E_8} = (\langle u \rangle, \langle v \rangle)^G$ and $|(\langle u \rangle, \langle v \rangle)^{E_8}| = 8$, it suffices to show that $G_{\langle u \rangle \langle v \rangle} = 1$. Pick an arbitrary $g \in G_{\langle u \rangle \langle v \rangle}$. First observe that all 2-orbits arising from the action of E_8 on the fibre set $\mathcal{F}(S)$ are thin, and hence this is also true of the action of G on $\mathcal{F}(S)$. Thus g fixes Ω_u and Ω_v pointwise. Now fix an arbitrary $w \in E_8 \setminus \langle u, v \rangle$, and choose an arbitrary pair of distinct points $x + \langle w \rangle, y + \langle w \rangle \in \Omega_w$.

If we assume $(x + \langle w \rangle)^g = y + \langle w \rangle$ for some $g \in G$, then it follows that each of the pairs $(x + \langle w \rangle, w + \langle u \rangle)$, $(y + \langle w \rangle, w + \langle u \rangle)$ and $(x + \langle w \rangle, w + \langle v \rangle)$, $(y + \langle w \rangle, w + \langle v \rangle)$ lie in a common 2-orbit of G (and hence also of E_8). However, a direct check shows that this is not the case. Hence $(x + \langle w \rangle)^g \neq y + \langle w \rangle$, which implies that g acts trivially on Ω_w.

The above argument proves that g acts trivially on all fibres with the possible exception of Ω_{u+v}. However, replacing $\langle u \rangle$ and $\langle v \rangle$ by some $\langle w \rangle$ and $\langle x \rangle$ for which $u + v \notin \langle x, w \rangle$, we may argue as above to establish that g acts trivially on Ω_{u+v} as well. $\square$

Proposition 4.2. *The centralizer $C_{\mathsf{Sym}(\Omega)}(E_8)$ of (E_8, Ω) in the symmetric group* $\mathrm{Sym}(\Omega)$ *is an elementary abelian group of order* 2^{14}.

Proof. A rather straightforward argument shows that the centralizer of a permutation group consists of those permutations which may be presented as a union of thin 2-orbits of that group. Combining this with the description of 2-orbits given in Example 1, we obtain that each permutation from $C_{\mathsf{Sym}(\Omega)}(E_8)$ is a union $\cup_{\theta \in \Theta} \widehat{\theta}$, where Θ is an arbitrary transversal of the sets $\{S_{\Omega_i}\}_{i=0}^{7}$. This yields the direct product $(\mathbb{Z}_2 \times \mathbb{Z}_2)^7 \cong E_{2^{14}}$. $\square$

Proposition 4.3. $C\mathrm{Aut}(S) = N_{\mathsf{Sym}(\Omega)}(E_8)$.

Proof. This follows immediately, since S is Schurian and (E_8, Ω) is 2-closed. $\square$

Proposition 4.4. $C\mathrm{Aut}(S) \cong E_{2^{14}} \rtimes GL(3, 2)$*, a group of order* $2^{17} \cdot 3 \cdot 7$.

Proof. The group $\mathsf{Aut}(E_8) \cong GL(3, 2)$ acts on Ω in a natural way. Thus the permutation subgroup $G \leq \mathsf{Sym}(\Omega)$ induced by this action normalizes E_8. As G permutes the sets Ω_v faithfully ($v \in E_8 \setminus \{0\}$), we have $G \cap C_{\mathsf{Sym}(\Omega)}(E_8) = 1$. Taking this together with the embedding $N_{\mathsf{Sym}(\Omega)}(E_8)/C_{\mathsf{Sym}(\Omega)}(E_8) \hookrightarrow \mathsf{Aut}(E_8)$, we obtain that $N_{\mathsf{Sym}(\Omega)}(E_8) = C_{\mathsf{Sym}(\Omega)}(E_8) \rtimes G$. To complete the proof, now simply apply Propositions 4.2 and 4.3. $\square$

Corollary 4.5. $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ *is a group of order* $2^{14} \cdot 3 \cdot 7$.

Proof. For each $g \in N_{\mathsf{Sym}(\Omega)}(E_8)$, denote by $\widetilde{g}$ the algebraic automorphism of S induced by g. Then the mapping $g \mapsto \widetilde{g}$ is a homomorphism, and its kernel is E_8. $\square$

4.2. AAut(S)

Recall that the group $A\mathrm{Aut}(S)$ of algebraic automorphisms of (Ω, S) acts as a permutation group on S. According to Example 1 we may write $S = S_1 \cup S_2$, where S_1 consists of the 28 relations of the form $\widehat{\alpha}$ (within fibres) and S_2 is comprised of the remaining 84 relations (across fibres). Clearly, S_1 and S_2 are invariant under $A\mathrm{Aut}(S)$.

In what follows we consider the action of $A\mathrm{Aut}(S)$ restricted to the set S_2. With respect to this action, S_2 may be further partitioned into seven subsets of the form

$$S_2^t := \{s_{ij}^0, s_{ij}^1 \mid i, j \text{ are incident to line } \ell_t, i \neq j\}, \quad 1 \le t \le 7$$

in accordance with Example 2. These subsets are in one–one correspondence with the lines of the Fano plane, and each subset consists of 12 relations. (Indeed l_t, $1 \le t \le 7$, represents the color of a unique line in the Fano plane. For any fixed t, there are $\binom{3}{2}$ choices for a pair of fibres Δ_{j_t}, Δ_{k_t} which admit a relation of color l_t, and for each such choice of fibre pair we get four relations.) We now have

Proposition 4.6. (a) *The action* $(A\mathrm{Aut}(S), S_2)$ *is faithful and transitive.*
(b) *The action* $(A\mathrm{Aut}(S), S_2)$ *is imprimitive, with imprimitivity system given by* $\mathcal{I} := \{S_2^1, S_2^2, \ldots, S_2^7\}$.
(c) *The kernel of the action* $(A\mathrm{Aut}(S), \mathcal{I})$ *is an elementary abelian group of order* 2^{14}.
(d) *The image of the action* $(A\mathrm{Aut}(S), \mathcal{I})$ *is a group isomorphic to* $GL(3, 2)$.

Proof. It suffices to prove part (c), as all remaining parts of the proposition follow rather easily from it. Clearly the kernel of the action $(A\mathrm{Aut}(S), \mathcal{I})$ has the form K^7, where K consists of those algebraic automorphisms which permute relations within a fixed imprimitivity block. We describe these automorphisms presently.

For each $u, v \in E_8 \setminus \{0\}$ with $u \neq v$, we define q_{uv} to be the involutory permutation of S which moves the eight indicated relations in the manner specified, while fixing all others[4]:

$$b_{uv}^1 \leftrightarrow b_{uv}^0, \qquad b_{vu}^1 \leftrightarrow b_{vu}^0, \qquad b_{u\,v+u}^1 \leftrightarrow b_{u\,v+u}^0, \qquad b_{v\,v+u}^1 \leftrightarrow b_{v\,v+u}^0.$$

Clearly q_{uv} is in the kernel of $(A\mathrm{Aut}(S), \mathcal{I})$. Furthermore, letting $l = \{u, v, w\} \subset E_8 \setminus \{0\}$ be the line of the Fano plane which corresponds to our fixed imprimitivity block above, we get that q_{uv}, q_{vw} and q_{wu} are the only nontrivial algebraic automorphisms in K. Hence K is the elementary abelian group of order 4, and (c) follows. $\square$

Corollary 4.7. $A\mathrm{Aut}(S) \cong E_{2^{14}} \rtimes GL(3, 2)$.

Remark. We refer to the automorphisms q_{uv} which appear in the proof of Proposition 4.6 as *elementary involutory algebraic automorphisms*.

4.3. Proper algebraic automorphisms

Corollaries 4.5 and 4.7 together imply that $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ embeds in $A\mathrm{Aut}(S)$ as a (non-normal) subgroup of index 8. As a consequence, there exist proper algebraic automorphisms of $AP(2)$. In fact, as may be revealed upon closer examination, the elementary involutory permutations q_{uv} of Section 4.2 are examples of such automorphisms.

[4] A similar algebraic automorphism was defined in [11].

Let now S' be a fusion of S, and let φ be an algebraic automorphism. Then $S'' = (S')^{\varphi}$ is again a fusion of S. However, when φ is proper it is no longer obligatory that S' and S'' be combinatorially isomorphic. In this case, we refer to the two fusions S' and S'' as twins.

Examples of algebraic twins are presented in Section 6. In each case, our methodology involves first the identification of a Schurian scheme obtained via a certain merging of classes, and then the realization of its non-Schurian twin via application of a suitable proper algebraic automorphism. Generally speaking, this automorphism is "hidden from view", its detection being facilitated by computer.

5. Homogeneous fusions/mergings in $AP(2)$

We continue to denote by S the set of relations of the half-homogeneous coherent configuration $AP(2)$. As we are interested in mergings of $AP(2)$ which lead to association schemes, we may consider the equivalent problem of identifying those fusions of S which are homogeneous. Using COCO-II,[5] we were able to enumerate all such fusions. To describe the results let us denote by $\mathfrak{F}$ the set of all homogeneous fusions of S. We say that two fusions $T, R \in \mathfrak{F}$ are equivalent if they are (combinatorially) isomorphic as coherent configurations. A complete enumeration using COCO-II gave us 47 equivalence classes of homogeneous fusions. These results were verified independently by checking the catalogue [18]. We next identified each equivalence class with a unique "representative" association scheme. Accordingly, we henceforth denote our equivalence classes by $\mathfrak{F}_i$, where i is the index of the chosen representative, with numeration as in [18]. The resulting list takes the form $\{\mathfrak{F}_i\}_{i \in \mathcal{I}_S \cup \mathcal{I}_N}$, where

$$\begin{aligned}\mathcal{I}_S &= \{1, 2, 3, 6, 10, 55, 72, 74, 76, 79, 87, 91, 95, 109, 111, 113, 114,\\ &\quad 138, 145, 147, 155, 175\},\\ \mathcal{I}_N &= \{7, 17, 20, 22, 24, 25, 26, 36, 38, 39, 40, 56, 57, 58, 59, 60, 61, 62,\\ &\quad 63, 64, 65, 66, 75, 110, 176\}.\end{aligned}$$

Note that the first set consists of Schurian fusions while the second one contains the non-Schurian ones.

It is important to note that each equivalence class $\mathfrak{F}_i$ is comprised of not a single $C\mathrm{Aut}(S)$-orbit but a union of them. In fact, working with $C\mathrm{Aut}(S)$-orbits in place of equivalence classes has clear advantages, but unfortunately the number of such orbits is prohibitively large. In Fig. 1, we provide a Hasse diagram of the lattice of all equivalence classes of fusions. A node i in the diagram represents the equivalence class $\mathfrak{F}_i$. Two nodes i and j are joined by an edge if and only if there exist two fusions $R \in \mathfrak{F}_i$ and $T \in \mathfrak{F}_j$ for which $R \sqsubseteq T$. Solid nodes indicate Schurian schemes; hollow nodes are reserved for non-Schurian ones.

The reader will note in Fig. 1 the existence of two special nodes labeled A and B. This is done for purely esthetic reasons. Each of these nodes may be interpreted as a family of equivalence classes. More specifically, node A can be replaced by any of $\{56, 57, 58, 66\}$ and node B by any of $\{17, 20, 22, 24, 25, 26, 36, 38, 39, 40, 60, 61, 62, 63, 64, 65\}$. All indicated association schemes here are maximal, that is, none has a homogeneous fission which is a fusion of S. However, to indicate all such classes explicitly would render our diagram quite unreadable.

Let us say that two equivalence classes $\mathfrak{F}_i$ and $\mathfrak{F}_j$ are *algebraic twins* if there exist $R \in \mathfrak{F}_i$ and $T \in \mathfrak{F}_j$ which are algebraically isomorphic, i.e. $R = T^{\varphi}$ for some $\varphi \in A\mathrm{Aut}(S)$. The property of

[5] This is an experimental version of COCO which is in development and intended as a future share package of GAP. The design team currently consists of C. Pech, I.A. Faradžev, M. Klin and S. Reichard.

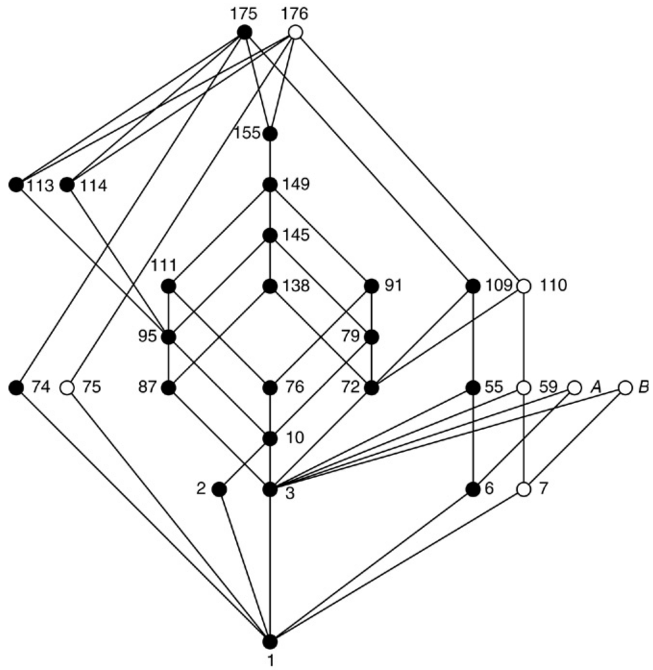

Fig. 1. Hasse diagram of the lattice of discovered fusions of S.

being algebraic twins induces an equivalence relation on the set $\{\mathfrak{F}_i\}_{i\in\mathcal{I}_S\cup\mathcal{I}_N}$ despite the fact that $A\mathrm{Aut}(S)$ does not act on these classes. Below we give all algebraic twin classes of size at least 2:

$$\{6,7\}, \{17,55,59\}, \{20,36,56,60\}, \{22,26,39,58,62\}, \{24,25,38,40,57,61\},$$
$$\{63,64,65,66\}, \{74,75\}\{109,110\}, \{175,176\}.$$

Note that some classes contain both Schurian and non-Schurian fusions.

A crucial observation here is that the property of being algebraic twins does not respect the order relation defined on the classes $\mathfrak{F}_i$. This is in part because each class $\mathfrak{F}_i$ is a union of $C\mathrm{Aut}(S)$-orbits rather than just a single such orbit. For example, $\mathfrak{F}_{55}$ and $\mathfrak{F}_{17}$ are algebraic twins despite the fact that $\mathfrak{F}_{17}$ is maximal while $\mathfrak{F}_{55}$ is not. We explain this phenomenon presently.

The class $\mathfrak{F}_{17}$ is actually a single $C\mathrm{Aut}(S)$-orbit, while each of $\mathfrak{F}_{55}$, $\mathfrak{F}_{59}$ is a union of two such orbits, say $\mathfrak{F}_i = \mathfrak{F}'_i \cup \mathfrak{F}''_i$, $i \in \{55, 59\}$. The orbits $\mathfrak{F}'_{55}$, $\mathfrak{F}'_{59}$ and $\mathfrak{F}_{17}$ form a single $A\mathrm{Aut}(S)$-orbit which is maximal in $\mathfrak{F}$. In contrast, the orbits $\mathfrak{F}''_{55}$ and $\mathfrak{F}''_{59}$ form an $A\mathrm{Aut}(S)$-orbit which is not maximal. (Note that $\mathfrak{F}_{55}$, $\mathfrak{F}_{59}$ and $\mathfrak{F}_{17}$ are pairwise algebraic twins by our definition.) Thus, $\mathfrak{F}_{55}$ is covered in Hasse diagram of our lattice despite the fact that members of $\mathfrak{F}''_{55}$ are actually maximal fusions of S.

6. Analysis of fusions; some examples of twins

The main goal of this section is to explore the occurrence of twins in a somewhat detailed manner. While we normally prefer to identify the Schurian member of a twin via the method of Galois correspondence (or some similar theoretical means), its non-Schurian twin is most often described via a suitable algebraic automorphism found with the aid of a computer. Once more, S denotes the set of relations of $AP(2)$.

6.1. Twins #175 and #176

This pair of twins is comprised of the only quasithin schemes on 28 points which have 15 classes. Scheme #175 is Schurian; indeed it is the scheme of 2-orbits of $AGL(1, 8)$ acting on Ω. Scheme #176 is non-Schurian with automorphism group E_8. Both schemes were discussed in detail in Section 3.2, where it was shown that they arise via mergings of $AP(2)$; see Corollary 3.4.

Sylow 7-subgroups of $A\mathrm{Aut}(S)$ are cyclic of order 7. Let us denote by Syl(7) the set of all such groups. As $CA/A := C\mathrm{Aut}(S)/\mathrm{Aut}(S)$ embeds in $A\mathrm{Aut}(S)$, we may consider the action of CA/A on Syl(7) via conjugation. We find that there are exactly two orbits in this action: one orbit consists of Sylow 7-subgroups of the embedded image of CA/A inside $A\mathrm{Aut}(S)$; the remaining orbit lies wholly outside this embedded image. The fusion of S with respect to a member of the first orbit gives us a Schurian scheme, and hence it is #175; the fusion with respect to a member of the second orbit yields #176. Since all Sylow 7-subgroups are evidently conjugate in $A\mathrm{Aut}(S)$, there exists an algebraic automorphism of $AP(2)$ which maps #175 onto #176. Thus the two schemes are algebraically, but not combinatorially, isomorphic.

6.2. Twins #74 and #75

Scheme #74 is the famous pseudocyclic scheme on 28 points discovered by Mathon [31]. (We refer the reader to [5] for a discussion on pseudocyclic schemes.) The most direct way to define it is in group theoretic terms. The group $SL(2, 8)$ contains a dihedral subgroup D_9 of order 18. Therefore the permutation representation of $SL(2, 8)$ on the cosets of D_9 has degree 28. This action is faithful and has three nontrivial 2-orbits, each symmetric of valency 9. The affine group $AGL(1, 8)$ is contained in $SL(2, 8)$ as the subgroup of upper-triangular matrices. A simple counting argument reveals that $|AGL(1, 8) \cap D_9| = 2$ and $AGL(1, 8)D_9 = SL(2, 8)$. Thus, $AGL(1, 8)$ acts transitively on the 28 cosets of D_9, and a stabilizer in this action is a subgroup $H \leq AGL(1, 8)$ of order 2. (Note that all subgroups of order 2 in $AGL(1, 8)$ are conjugate.) Therefore, scheme #74 is a Schurian fusion of the scheme $AGL(1, 8)/\!\!/H$ of 2-orbits of $AGL(1, 8)$ acting on the cosets of H, which we know to be isomorphic to the scheme #175. As #175 and #176 are twins, applying the evident algebraic automorphism to #74 produces *its* twin; it is the scheme #75 discovered by Hollmann. Scheme #75 is non-Schurian. Computing its automorphism group with the aid of COCO-II, we conclude that it is elementary abelian of order 8.

6.3. Twins #109 and #110

Here, for the first time in this paper, we are faced with a pleasant surprise: the existence of a pair of twins which, in evident form, was not previously known to the mathematical audience.

Both schemes appear as algebraic fusions with respect to different groups of algebraic automorphisms. As we have already observed, there are two $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$-conjugacy classes of Sylow 7-subgroups of $A\mathrm{Aut}(S)$; consequently, there are two such classes of Sylow normalizers. We denote representatives of these classes by N_1 and N_2. Note that these normalizers are conjugate in $A\mathrm{Aut}(S)$, each isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

One of these Sylow normalizers, say N_1, is contained in $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$; hence the algebraic fusion with respect to N_1 is Schurian. This is the scheme #109. Its automorphism group, computed with the aid of COCO-II, is isomorphic to $A\Gamma L(1, 8)$. The second algebraic fusion (i.e., with respect to the group N_2) is the non-Schurian scheme #110. Its automorphism group is isomorphic to $A_4 \times \mathbb{Z}_2$, again courtesy of COCO-II.

6.4. Twins #6 and #7

Here is the most famous pair of twins — the triangular graph $T(8)$ and one of the Chang graphs. Recall that the triangular graph $T(n)$ has as its vertices all the two-element subsets of an n-element set, with two vertices adjacent if they intersect in one element. $T(n)$ is a strongly regular graph, and its automorphism group coincides with the symmetric group S_n acting on the set $\mathcal{P}_2(n)$ of two-element subsets of $\{1, \ldots, n\}$. Graph $T(n)$ and its complement together form a two-class Schurian association scheme. Thus #6 is a Schurian fusion of our half-homogeneous coherent configuration $AP(2)$. The automorphism group of #6 coincides with S_8 in its action on two-element subsets of an eight-element set.

Scheme #7 has the same parameters as scheme #6, since it is obtained from #6 by applying an algebraic automorphism. Therefore one of its basic graphs is a strongly regular graph with the same parameters as $T(8)$. As mentioned in the Introduction, $T(n)$ is uniquely determined by its parameters when $n \neq 8$, and when $n = 8$ there are three additional strongly regular graphs with the same parameters as $T(8)$, called the Chang graphs. Each Chang graph may be obtained from $T(8)$ via the method of *Seidel's switching*.[6]

Define the following subsets of $\mathcal{P}_2(8)$:

$$\Sigma_1 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\},$$
$$\Sigma_2 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,1\}\},$$
$$\Sigma_3 = \{\{1,2\},\{2,3\},\{3,1\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,4\}\}.$$

Then the graphs obtained from $T(8)$ by switching with respect to Σ_i, $i = 1, 2, 3$, are the strongly regular Chang graphs. We denote the Chang graph obtained by switching with respect to Σ_i as T_{Σ_i}.

Computer computations reveal that among the Chang graphs, only T_{Σ_1} admits the appropriate elementary abelian subgroup of order 8. Thus, scheme #7 is formed by T_{Σ_1} and its complement. The automorphism group of #7, computed by COCO, is $S_2 \wr S_4$ acting on two-element subsets of an eight-element set.

6.5. Twins #55 and #59, and fusions from $\mathfrak{F}_A \cup \mathfrak{F}_B$

We consider these fusions together because they may be described in a unified manner. In fact, we shall soon see that each may be obtained by removing a certain type of spread from one of the respective complements $\overline{T(8)}$, $\overline{T_{\Sigma_1}}$ of the graphs $T(8)$, T_{Σ_1}. Recall that a *spread* in a strongly regular graph is a partition of its vertices into a disjoint union of Delsarte cliques.[7]

Haemers and Tonchev proved in [17] that a strongly regular graph with a spread gives rise to an imprimitive three-class association scheme. The relations of this scheme correspond to (1) the spread, (2) the strongly regular graph minus the spread, and (3) the complementary strongly regular graph. Let us refer to the first of these as the *spread relation* of the scheme.

As a consequence, given any scheme $S' \in \mathfrak{F}_6 \cup \mathfrak{F}_7$ removing a spread from the appropriate graph (i.e., from $\overline{T(8)}$ if $S' \in \mathfrak{F}_6$, and from $\overline{T_{\Sigma_1}}$ if $S' \in \mathfrak{F}_7$) yields a three-class fission of S'. If this spread is moreover invariant with respect to the action of E_8 on Ω, then S' is also a fusion of S; in fact $S' \in \mathfrak{F}_{55} \cup \mathfrak{F}_{59} \cup \mathfrak{F}_A \cup \mathfrak{F}_B$.

6 See [5] for a detailed description of this method.

7 A *Delsarte clique* is a clique of size meeting the Delsarte bound $1 - k/s$, where k is the valency and s is the smallest eigenvalue of an ambient strongly regular graph.

We now describe how one detects such spreads in the graphs $\overline{T(8)}$ and $\overline{T_{\Sigma_1}}$. Clearly, a spread is a partition of the 28 points of Ω into seven cells of size 4; hence the spread relation must be a symmetric relation consisting of 84 arcs which is moreover obtained via a merging of relations from $AP(2)$.

A natural choice of spread is the set of fibres of $AP(2)$. Setting $\Theta_j = (\Delta_j)^2 \setminus i_{\Delta_j}$, we see that the spread relation here is $\Theta_1 \cup \Theta_2 \cup \cdots \cup \Theta_7$. This is clearly a merging of relations from $AP(2)$ since $\Theta_j = s_{1j} \cup s_{2j} \cup s_{3j}$ (cf. Example 2). Removal of this spread from $\overline{T(8)}$ gives the scheme #55, while its removal from $\overline{T_{\Sigma_1}}$ yields #59. As these are the most symmetric schemes we shall encounter in this subsection, we will discuss them in more detail a bit later on. But first let us look at how some less evident spreads may be formed, and what types of schemes result from their removal.

Referring to Example 2, pick two arbitrary fibres, say Δ_1 and Δ_2, and define the relation

$$s_{12}^0 \cup s_{21}^0 \cup s_{11} \cup s_{12} \cup \Theta_3 \cup \cdots \cup \Theta_7.$$

This is a symmetric relation with 84 arcs, which is clearly obtained via a merging of relations from $AP(2)$. Hence it is a candidate for a spread relation. What must be checked is whether or not this relation lives inside the appropriate complement graph. When inside of $\overline{T(8)}$ we get a scheme from $\mathfrak{F}_A$; when inside $\overline{T_{\Sigma_1}}$ a scheme from $\mathfrak{F}_B$ is obtained. In either case, the corresponding spread is given (cf. Example 2) by

$$\{a_1, b_1, c_2, d_2\}, \{a_2, b_2, c_1, d_1\}, \Delta_3, \ldots, \Delta_7.$$

Another feasible candidate for the spread relation is

$$s_{12}^1 \cup s_{21}^1 \cup s_{11} \cup s_{12} \cup \Theta_3 \cup \cdots \cup \Theta_7,$$

with corresponding spread

$$\{a_1, a_2, b_1, b_2\}, \{c_1, c_2, d_1, d_2\}, \Delta_3, \ldots, \Delta_7.$$

In this manner, we produce $2 \cdot \binom{7}{2} = 42$ candidates for spreads by varying our choice of fibre pair. Moreover, this construction can be generalized by switching not just within a single fibre pair, but simultaneously within each of two or three fibre pairs. In every such case we get a scheme from $\mathfrak{F}_A$ whenever the spread is actual in $\overline{T(8)}$, and one from $\mathfrak{F}_B$ when it is realized in $\overline{T_{\Sigma_1}}$. One particular instance of the latter case occurs for the scheme #17, which can be so obtained in different ways (i.e., via removal of inequivalent spreads with respect to $C\mathrm{Aut}(S)$). We emphasize this fact because schemes #55, #59 and #17 are actually triplets!

As promised, we now return to schemes #55 and #59 for a closer examination. Both are algebraic fusions. The group $C\mathrm{Aut}(S)$ contains a subgroup H which is isomorphic to $GL(3, 2)$. Group H acts transitively on the set of fibres; thus the algebraic fusion with respect to H is homogeneous. As $H \leq C\mathrm{Aut}(S)$, this fusion is Schurian. Finally, since H has three orbits in its action on the set of non-diagonal binary relations of $AP(2)$ the resulting fusion has three classes. This uniquely identifies it as #55. Its automorphism group, computed via COCO, is $AGL(3, 2)$.

Consider now the projection $C\mathrm{Aut}(S) \to C\mathrm{Aut}(S)/\mathrm{Aut}(S)$. As $GL(3, 2)$ is a simple group, the image of H under this projection is again isomorphic to $GL(3, 2)$. As a consequence, we may identify H as a subgroup of $A\mathrm{Aut}(S)$. Conjugating H by a suitable element of $A\mathrm{Aut}(S)$, we obtain an embedding of $GL(3, 2)$ which is *not* contained in $C\mathrm{Aut}(S)/\mathrm{Aut}(S)$. Merging classes of $AP(2)$ with respect to this copy of $GL(3, 2)$ yields a non-Schurian twin of #55, namely the scheme #59. The automorphism group of #59 has order 192 and is isomorphic to $(S_2 \wr S_4)^{\text{pos}}$, the group of rotations of the four-dimensional cube.

Remark. Both association schemes #55 and #59 are briefly mentioned in [9] with a credit given to E. Spence (among many other schemes with the same parameters).

7. Concluding remarks

Here we discuss our project in a more general context, adding a number of significant historical and bibliographical comments. We also compare it to a few similar projects recently undertaken by some of the authors, outlining some attractive perspectives for combining and extending our methodology and results.

7.1. Algebraic and combinatorial isomorphisms and automorphisms

Combinatorial isomorphisms of coherent configurations were introduced independently by D.G. Higman and B. Weisfeiler et al. In his seminal paper [19], Higman speaks of automorphisms and strict automorphisms when referring to elements of $C\text{Aut}(S)$ and $\text{Aut}(S)$, respectively. In [41], colored automorphisms are referred to as weak automorphisms.

Algebraic isomorphisms and automorphisms were introduced in [41] under the respective names of weak equivalence and weak natural equivalence. Note, however, that for the particular case of commutative association schemes the notion of an algebraic isomorphism goes back to Bose and Mesner [4] (thus justifying Godsil's use of the term Bose–Mesner automorphism in [16]).

Practical use of algebraic automorphisms goes back to Faradžev and his colleagues (see Section 2.7.6 of [13]), where the notion appears as a necessary condition for the existence of combinatorial isomorphism. In evident form, algebraic automorphisms were first exploited by C. Pech (1997–98), who elaborated a first efficient version of the normalizer algorithm for 2-closed permutation groups. Though the algorithm was never formally published, it was used very essentially by Fiedler in his Master's thesis [14]. Recently, similar algorithmic techniques have been developed by Miyamoto, e.g., see [34].

Regarding algebraic fusion, the observation made in the opening paragraph of Section 2.6 is of folklore nature. Its particular case for subgroups of $C\text{Aut}(S)$ appears, for example, in [20]. Note that the notion of symmetrization of a commutative association scheme is a particular case of algebraic fusion.

7.2. WFDF configurations

As defined in this paper, the notion of a half-homogeneous coherent configuration is of quite general nature. Although in theory one may conduct our type of investigation starting from a general such object, practical experience has taught us that a certain more specialized object is ideally suited for our purposes. We speak now of a Wallis–Fon-Der-Flaass configuration (WFDF configuration, for short).

In evident form, the notion of a WFDF configuration initially appeared in [28], following the ideas of Muzychuk. Roughly speaking, its initial ingredients are an affine plane of order n and a linear space of order m such that each point of the linear space is incident to at most $n+1$ lines. As a result, we get a coherent configuration $S = (V, \mathcal{R})$ with point set V of cardinality n^2m, with $|\mathcal{R}| = m(n+2m)$ relations, and with m fibres.

Two particular cases of the above lead to the families $AP(n)$ (used in the present investigation, when $n = 2$) and $AK(n)$ (treated in [28], again for $n = 2$). In the former construction the role of the linear space is fulfilled by a projective plane of order n (hence the symbol "P" in $AP(n)$) while in the latter construction this role is relegated to the complete graph K_{n+2} on $n+2$ vertices (explaining the symbol "K"). Evidently, in both cases the symbol "A" refers to "affine".

Regarding the aforementioned treatment in [28] of the WFDF configuration $AK(2)$, the methodology essentially mirrors the one used in this paper for $AP(2)$. In spirit, the results are similar as well: a number of famous and interesting combinatorial objects on 16 points, including the Shrikhande graph, are now given new interpretations through the aid of algebraic mergings and the notion of twins.

It happens that these specialized classes of structures themselves suggest additional directions for possible generalization. For example, one may consider the broader class of resolvable designs in place of affine planes. In a different vein, one may relax the rules of construction, putatively leading to objects more general than coherent configurations, for example color graphs.

In fact one such line of generalization has already been investigated. We speak now of a recent paper of Muzychuk [36], in which he describes a prolific construction of strongly regular graphs based on an approach very close to the one employed by us in this paper. Although coherent configurations are not explicitly used in [36], the attentive reader will certainly recognize a common theoretical thread in the two papers.

Conceptually speaking, WFDF configurations are still in their initial stage of development, which explains why we decided to restrict our consideration to $AP(2)$ intentionally avoiding explicit attempts at generalization. Nevertheless, it is clear to us that we are in the vicinity of some very promising attempts toward a more general, systematic methodology.

7.3. *Algebraic twins*

As defined in this paper, the notion of algebraic twins begs for further elucidation. To accomplish this, it is most convenient to elaborate two variations of this notion: *absolute* and *relative*.

Coherent configurations S_1 and S_2 are called *absolute algebraic twins* if they are non-isomorphic yet have isomorphic tensors of structure constants. This notion has existed for many years. The smallest known example consists of the two strongly regular graphs with parameter set $(v, k, l, \lambda, \mu) = (16, 6, 9, 2, 2)$, namely the line graph $L_2(4)$ and the Shrikhande graph Sh. (In fact, the existence of this pair of absolute algebraic twins was detected for the first time by Mesner in his seminal Ph.D. thesis [33].)

Assume now that S_1 and S_2 are non-isomorphic coherent configurations which occur as fusions of the same coherent configuration S. Assume further that $(S_1)^g = S_2$ for some $g \in A\mathrm{Aut}(S)$. Then we say S_1 and S_2 are *S-relative algebraic twins*. In this terminology, one may say that our present paper is devoted to the investigation of $AP(2)$-relative algebraic twins. We gratuitously observe that relative algebraic twins are necessarily absolute.

Let us refer to algebraic twins as *relative* if they are S-relative for some coherent configuration S. Then one may ask whether or not every pair of absolute twins is in fact relative. We feel this is an interesting problem, especially since it lays groundwork for a theoretical formulation of "degree of non-isomorphism". Our basic premise is simple: a pair of absolute (but non-relative) twins may be regarded as "further from isomorphism" than a pair of relative twins. Note that a similar notion already exists for finite groups.

We believe that the notion (and earliest examples) of relative twins was initiated by members of our scientific group. In [27] one encounters a pair of relative twins consisting of strongly regular graphs on 36 points which satisfy the four-vertex condition (in the sense of D.G. Higman). Unfortunately, the presentation in this paper does not allow the reader to identify either the unifying configuration S or the linking automorphism $g \in A\mathrm{Aut}(S)$. From this point of view our present exposition is certainly preferable.

Below we discuss one additional open problem, as well as prospects for its resolution. Namely, we ask whether or not it is possible to exhibit a family of embedded pairs of relative twins in an infinite series of similar objects.

7.4. Pseudocyclic association schemes

We again refer the reader to [5] for definitions, properties and examples of pseudocyclic association schemes, which appear as a generalization of cyclotomic association schemes.

In fact, not much is known about pseudocyclic association schemes with at least three classes which are not cyclotomic. A classical infinite series of such objects was constructed by Mathon in [31] with the aid of conics in $PG(2, q)$; e.g., see [5] for their brief description.

A purely group theoretic explanation of the existence of the Mathon schemes may be found in [38], though it appears that this detailed and quite beautiful source remains unnoticed by many experts in algebraic combinatorics. The presentation is based on the use of a classical exceptional isomorphism between $PSL(2, q)$ and $PO(3, q)$. Consider the action of $G = PSL(2, q)$, $q = 2^m$, on the cosets of its subgroup D_{q+1} (the dihedral group of order $2(q + 1)$), which is equivalent to the action of G, via conjugation, on its Singer cycles of length $q + 1$. It turns out that this is a generously transitive action of rank 2^{m-1} and degree $2^{m-1}(2^m - 1)$; hence it gives rise to a symmetric association scheme with $2^{m-1} - 1$ classes of valency $2^m + 1$. In fact, such schemes are pseudocyclic. (An independent proof of this fact may be found in the recent paper [26]; however no citation to [38] occurs there.)

We may enlarge the group $G = PSL(2, q)$ by the Frobenius automorphism, thus obtaining the group $\widehat{G} = P\Sigma O(3, q)$ of order $m(q + 1)q(q - 1)$. Group $\widehat{G}$ extends the action of G on the set of Singer cycles, resulting in a merging of classes in the corresponding Mathon scheme. In general this merging does not lead to a pseudocyclic association scheme; for example in [38] it is shown that when $m = 4$ one obtains a three-class fusion on 120 points with the valencies 17, 34, 68. However, when $m = p$ is an odd prime it is always the case that the resulting fusion is pseudocyclic with valencies $(2^p + 1)p$. This quite remarkable fact is mentioned without proof in [5,38] but may be easily verified using an algebraic automorphism of the Mathon scheme induced from the Frobenius automorphism of the field of 2^p elements. (Note that a rather involved proof of this fact is given in [26].)

Let us have a closer look at the two initial cases.

7.4.1. $q = 2^3 = 8$

Here, the Mathon scheme already has three classes and is exactly our scheme #74 of Section 6.2. Note that a self-contained elementary treatment of this scheme will also appear in [30].

7.4.2. $q = 2^5 = 32$

Here the Mathon scheme S_1 has 15 classes of valency 33 while its pseudocyclic fusion S_1' has three classes of valency 165. An interesting question is whether we can obtain S_1' (and possibly S_1 as well) as a fusion of a suitable half-homogeneous configuration S on 496 points. Here, two options for S seem particularly attractive to us.

On one hand, we may consider a certain generalization of an WFDF configuration of type $AP(n)$ obtained by replacing the affine plane by an affine 4-space over $GF(2)$, which has 16 points and 32 blocks of size 8. In the role of the linear space we take the projective Steiner triple system over $GF(2)$ with 31 points. The resulting configuration will have 31 fibres of size 16, each fibre inducing a thin scheme of order 16. This is our first candidate for S.

On the other hand, we may imitate our model in Example 1 as follows. Consider the one-dimensional affine group $AGL(1, 32)$ of order $31 \cdot 32$ and its subgroup H of order 2. The centralizer algebra of the action of $AGL(1, 32)$ on the cosets of H is a quasithin association scheme on 496 points. Consider now a subgroup $K \leq AGL(1, 32)$ of order 32 acting on these cosets via restriction. Its centralizer algebra corresponds to another half-homogeneous coherent configuration with 31 fibres of size 16, this being our second candidate for S.

In fact, quite easy arguments establish that our two candidates for S are non-isomorphic. Thus in each case there is promise that an investigation of mergings and automorphisms may uncover a pair of S-relative algebraic twins. More explicitly, supposing that S_1 (or S_1') is a fusion of one or another candidate for S, then it would not be unrealistic to anticipate the existence of a twin S_2 (or S_2') obtained as another fusion of S with linking automorphism $S_1 \mapsto S_2$ (or $S_1' \mapsto S_2'$) coming from $A\mathrm{Aut}(S)$.

Moreover, the existence of a pair of S-relative twins would indeed promote hope that one could, in a similar manner, embed successive members of the Mathon infinite series (or its mergings) as $q = 2^p$ (p prime) grows.

Resolution of both problems will require essential innovations in the performance design of COCO-II. We are considering such developments at present and hope to make a breakthrough in the near future.

Acknowledgments

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Appendix A

We here provide the main computational data used in the preparation of this paper. All data were obtained through the use of GAP, GRAPE and an experimental version of COCO.

A.1. Initial group action and 2-orbits

We start with the following presentation of $E(8)$:
$E(8) = \langle \pi_1, \pi_2, \pi_3 \rangle$, where

$\pi_1 : [x, y, z] \mapsto [x, y, z] + [1, 0, 0]$,
$\pi_2 : [x, y, z] \mapsto [x, y, z] + [0, 1, 0]$,
$\pi_3 : [x, y, z] \mapsto [x, y, z] + [0, 0, 1]$.

The following mapping was used to induce the group in GAP:

1	[0, 0, 0]	5	[1, 0, 0]
2	[0, 0, 1]	6	[1, 0, 1]
3	[0, 1, 0]	7	[1, 1, 0]
4	[0, 1, 1]	8	[1, 1, 1]

With this mapping we get

$$E(8) = \langle (1, 5)(2, 6)(3, 7)(4, 8), (1, 3)(2, 4)(5, 7)(6, 8), (1, 2)(3, 4)(5, 6)(7, 8)\rangle.$$

Next, this group was induced on the two-element subsets of $\{1, 2, \ldots, 8\}$. The following numeration of subsets was used:

1	{1, 2}	8	{2, 3}	15	{3, 5}	22	{4, 8}
2	{1, 3}	9	{2, 4}	16	{3, 6}	23	{5, 6}
3	{1, 4}	10	{2, 5}	17	{3, 7}	24	{5, 7}
4	{1, 5}	11	{2, 6}	18	{3, 8}	25	{5, 8}
5	{1, 6}	12	{2, 7}	19	{4, 5}	26	{6, 7}
6	{1, 7}	13	{2, 8}	20	{4, 6}	27	{6, 8}
7	{1, 8}	14	{3, 4}	21	{4, 7}	28	{7, 8}

This mapping gives the following induced action:

$$\begin{aligned} E(8) = \langle &(1, 23)(2, 24)(3, 25)(5, 10)(6, 15)(7, 19)(8, 26)(9, 27)(12, 16)(13, 20)(14, 28)(18, 21), \\ &(1, 14)(3, 8)(4, 17)(5, 18)(6, 15)(7, 16)(10, 21)(11, 22)(12, 19)(13, 20)(23, 28)(25, 26), \\ &(2, 9)(3, 8)(4, 11)(5, 10)(6, 13)(7, 12)(15, 20)(16, 19)(17, 22)(18, 21)(24, 27)(25, 26)\rangle. \end{aligned}$$

The above action of the group $E(8)$ on 28 points has rank 112. As before, we denote the set of 2-orbits of this action by S.

Below we list representatives for each relation from S:

1	[1, 1]	17	[1, 5]	33	[2, 10]	49	[3, 6]	65	[4, 13]	81	[5, 7]	97	[6, 16]
2	[2, 2]	18	[1, 18]	34	[2, 6]	50	[3, 13]	66	[4, 7]	82	[5, 12]	98	[7, 1]
3	[3, 3]	19	[1, 6]	35	[2, 13]	51	[3, 7]	67	[4, 12]	83	[6, 1]	99	[7, 14]
4	[4, 4]	20	[1, 15]	36	[2, 7]	52	[3, 12]	68	[5, 1]	84	[6, 14]	100	[7, 2]
5	[5, 5]	21	[1, 7]	37	[2, 12]	53	[4, 1]	69	[5, 14]	85	[6, 2]	101	[7, 9]
6	[6, 6]	22	[1, 16]	38	[3, 1]	54	[4, 14]	70	[5, 2]	86	[6, 9]	102	[7, 3]
7	[7, 7]	23	[2, 1]	39	[3, 23]	55	[4, 2]	71	[5, 9]	87	[6, 3]	103	[7, 8]
8	[1, 14]	24	[2, 23]	40	[3, 2]	56	[4, 9]	72	[5, 3]	88	[6, 8]	104	[7, 4]
9	[1, 23]	25	[2, 9]	41	[3, 24]	57	[4, 3]	73	[5, 8]	89	[6, 4]	105	[7, 11]
10	[1, 28]	26	[2, 24]	42	[3, 8]	58	[4, 8]	74	[5, 4]	90	[6, 11]	106	[7, 5]
11	[1, 2]	27	[2, 27]	43	[3, 25]	59	[4, 11]	75	[5, 17]	91	[6, 5]	107	[7, 10]
12	[1, 24]	28	[2, 3]	44	[3, 26]	60	[4, 17]	76	[5, 10]	92	[6, 10]	108	[7, 6]
13	[1, 3]	29	[2, 25]	45	[3, 4]	61	[4, 22]	77	[5, 18]	93	[6, 13]	109	[7, 15]
14	[1, 25]	30	[2, 4]	46	[3, 11]	62	[4, 5]	78	[5, 21]	94	[6, 15]	110	[7, 12]
15	[1, 4]	31	[2, 11]	47	[3, 5]	63	[4, 18]	79	[5, 6]	95	[6, 20]	111	[7, 16]
16	[1, 17]	32	[2, 5]	48	[3, 10]	64	[4, 6]	80	[5, 13]	96	[6, 7]	112	[7, 19]

There are seven reflexive relations: 1, 2, 3, 4, 5, 6, 7.

There are 21 symmetric relations:
8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112.

The remaining 84 relations are antisymmetric. Below we give the 42 pairings of each such relation with its transpose:

11	23	23	11	38	13	53	15	68	17	83	19	98	21
12	24	24	12	39	14	54	16	69	18	84	20	99	22
13	38	28	40	40	28	55	30	70	32	85	34	100	36
14	39	29	41	41	29	56	31	71	33	86	35	101	37
15	53	30	55	45	57	57	45	72	47	87	49	102	51
16	54	31	56	46	58	58	46	73	48	88	50	103	52
17	68	32	70	47	72	62	74	74	62	89	64	104	66
18	69	33	71	48	73	63	75	75	63	90	65	105	67
19	83	34	85	49	87	64	89	79	91	91	79	106	81
20	84	35	86	50	88	65	90	80	92	92	80	107	82
21	98	36	100	51	102	66	104	81	106	96	108	108	96
22	99	37	101	52	103	67	105	82	107	97	109	109	97

In order to get a better overview, we give the following block distribution of relations:

{1, 8, 9, 10}	{11, 12}	{13, 14}	{15, 16}	{17, 18}	{19, 20}	{21, 22}
{23, 24}	{2, 25, 26, 27}	{28, 29}	{30, 31}	{32, 33}	{34, 35}	{36, 37}
{38, 39}	{40, 41}	{3, 42, 43, 44}	{45, 46}	{47, 48}	{49, 50}	{51, 52}
{53, 54}	{55, 56}	{57, 58}	{4, 59, 60, 61}	{62, 63}	{64, 65}	{66, 67}
{68, 69}	{70, 71}	{72, 73}	{74, 75}	{5, 76, 77, 78}	{79, 80}	{81, 82}
{83, 84}	{85, 86}	{87, 88}	{89, 90}	{91, 92}	{6, 93, 94, 95}	{96, 97}
{98, 99}	{100, 101}	{102, 103}	{104, 105}	{106, 107}	{108, 109}	{7, 110, 111, 112}

A.2. Homogeneous fusions

Here we enumerate all homogeneous fusions of S. We give one representative from each class $\mathfrak{F}_i$. The representatives are numbered according to the catalogue [18]. The listing is complete up to isomorphism.

S_{175} := {1, 2, 3, 4, 5, 6, 7}, {8, 26, 44, 61, 76, 93, 111}, {9, 25, 43, 60, 77, 95, 110}, {10, 27, 42, 59, 78, 94, 112}, {11, 30, 49, 57, 68, 96, 106}, {12, 31, 50, 58, 69, 97, 107}, {13, 34, 47, 66, 74, 83, 100}, {14, 35, 48, 67, 75, 84, 101}, {15, 28, 51, 64, 70, 91, 98}, {16, 29, 52, 65, 71, 92, 99}, {17, 23, 45, 55, 81, 87, 108}, {18, 24, 46, 56, 82, 88, 109}, {19, 36, 38, 62, 72, 85, 104}, {20, 37, 39, 63, 73, 86, 105}, {21, 32, 40, 53, 79, 89, 102}, {22, 33, 41, 54, 80, 90, 103}

S_{176} := {1, 2, 3, 4, 5, 6, 7}, {8, 26, 44, 61, 76, 93, 111}, {9, 25, 43, 60, 77, 95, 110}, {10, 27, 42, 59, 78, 94, 112}, {11, 30, 49, 57, 68, 96, 106}, {12, 31, 50, 58, 69, 97, 107}, {13, 34, 47, 67, 74, 83, 100}, {14, 35, 48, 66, 75, 84, 101}, {15, 28, 52, 64, 70, 91, 98}, {16, 29, 51, 65, 71, 92, 99}, {17, 23, 45, 55, 81, 87, 108}, {18, 24, 46, 56, 82, 88, 109}, {19, 36, 38, 62, 72, 85, 105}, {20, 37, 39, 63, 73, 86, 104}, {21, 32, 40, 53, 79, 89, 103}, {22, 33, 41, 54, 80, 90, 102}

S_{155} := {1, 2, 3, 4, 5, 6, 7}, {8, 26, 44, 61, 76, 93, 111}, {9, 25, 43, 60, 77, 95, 110}, {10, 27, 42, 59, 78, 94, 112}, {11, 30, 49, 57, 68, 96, 106}, {12, 31, 50, 58, 69, 97, 107}, {13, 14, 34, 35, 47, 48, 66, 67, 74, 75, 83, 84, 100, 101}, {15, 16, 28, 29, 51, 52, 64, 65, 70, 71, 91, 92, 98, 99}, {17, 23, 45, 55, 81, 87, 108}, {18, 24, 46, 56, 82, 88, 109}, {19, 20, 36, 37, 38, 39, 62, 63, 72, 73, 85, 86, 104, 105}, {21, 22, 32, 33, 40, 41, 53, 54, 79, 80, 89, 90, 102, 103}

S_{149} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 26, 43, 60, 77, 94, 111}, {10, 27, 44, 61, 78, 95, 112}, {11, 12, 28, 29, 45, 46, 62, 63, 79, 80, 96, 97, 98, 99}, {13, 14, 30, 31, 47, 48, 64, 65, 81, 82, 83, 84, 100, 101}, {15, 16, 32, 33, 49, 50, 66, 67, 68, 69, 85, 86, 102, 103}, {17, 18, 34, 35, 51, 52, 53, 54, 70, 71, 87, 88, 104, 105}, {19, 20, 36, 37, 38, 39, 55, 56, 72, 73, 89, 90, 106, 107}, {21, 22, 23, 24, 40, 41, 57, 58, 74, 75, 91, 92, 108, 109}

S_{145} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 10, 26, 27, 43, 44, 60, 61, 77, 78, 94, 95, 111, 112}, {11, 12, 28, 29, 45, 46, 62, 63, 79, 80, 96, 97, 98, 99}, {13, 14, 30, 31, 47, 48, 64, 65, 81, 82, 83, 84, 100, 101}, {15, 16, 32, 33, 49, 50, 66, 67, 68, 69, 85, 86, 102, 103}, {17, 18, 34, 35, 51, 52, 53, 54, 70, 71, 87, 88, 104, 105}, {19, 20, 36, 37, 38, 39, 55, 56, 72, 73, 89, 90, 106, 107}, {21, 22, 23, 24, 40, 41, 57, 58, 74, 75, 91, 92, 108, 109}

S_{138} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 12, 28, 29, 45, 46, 62, 63, 79, 80, 96, 97, 98, 99}, {13, 14, 30, 31, 47, 48, 64, 65, 81, 82, 83, 84, 100, 101}, {15, 16, 32, 33, 49, 50, 66, 67, 68, 69, 85, 86, 102, 103}, {17, 18, 34, 35, 51, 52, 53, 54, 70, 71, 87, 88, 104, 105}, {19, 20, 36, 37, 38, 39, 55, 56, 72, 73, 89, 90, 106, 107}, {21, 22, 23, 24, 40, 41, 57, 58, 74, 75, 91, 92, 108, 109}

S_{111} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 26, 43, 60, 77, 94, 111}, {10, 27, 44, 61, 78, 95, 112}, {11, 12, 13, 14, 23, 24, 30, 31, 38, 39, 47, 48, 55, 56, 64, 65, 72, 73, 81, 82, 89, 90, 96, 97, 106, 107, 108, 109}, {15, 16, 17, 18, 28, 29, 34, 35, 40, 41, 51, 52, 53, 54, 66, 67, 68, 69, 79, 80, 85, 86, 91, 92, 102, 103, 104, 105}, {19, 20, 21, 22, 32, 33, 36, 37, 45, 46, 49, 50, 57, 58, 62, 63, 70, 71, 74, 75, 83, 84, 87, 88, 98, 99, 100, 101}

S_{110} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 15, 19, 28, 30, 36, 38, 49, 51, 57, 62, 64, 68, 70, 73, 85, 92, 96, 98, 104, 106}, {12, 16, 20, 29, 31, 37, 39, 50, 52, 58, 63, 65, 69, 71, 72, 86, 91, 97, 99, 105, 107}, {13, 17, 21, 23, 32, 34, 40, 45, 48, 53, 55, 66, 74, 80, 81, 83, 87, 89, 100, 102, 108}, {14, 18, 22, 24, 33, 35, 41, 46, 47, 54, 56, 67, 75, 79, 82, 84, 88, 90, 101, 103, 109}

S_{91} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 26, 43, 60, 77, 94, 111}, {10, 27, 44, 61, 78, 95, 112}, {11, 12, 13, 14, 15, 16, 28, 29, 32, 33, 34, 35, 45, 46, 47, 48, 51, 52, 55, 56, 64, 65, 66, 67, 68, 69, 74, 75, 79, 80, 83, 84, 87, 88, 96, 97, 98, 99, 100, 101, 106, 107}, {17, 18, 19, 20, 21, 22, 23, 24, 30, 31, 36, 37, 38, 39, 40, 41, 49, 50, 53, 54, 57, 58, 62, 63, 70, 71, 72, 73, 81, 82, 85, 86, 89, 90, 91, 92, 102, 103, 104, 105, 108, 109}

S_{95} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 10, 26, 27, 43, 44, 60, 61, 77, 78, 94, 95, 111, 112}, {11, 12, 13, 14, 23, 24, 30, 31, 38, 39, 47, 48, 55, 56, 64, 65, 72, 73, 81, 82, 89, 90, 96, 97, 106, 107, 108, 109}, {15, 16, 17, 18, 28, 29, 34, 35, 40, 41, 51, 52, 53, 54, 66, 67, 68, 69, 79, 80, 85, 86, 91, 92, 102, 103, 104, 105}, {19, 20, 21, 22, 32, 33, 36, 37, 45, 46, 49, 50, 57, 58, 62, 63, 70, 71, 74, 75, 83, 84, 87, 88, 98, 99, 100, 101}

S_{109} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 15, 19, 28, 30, 36, 38, 49, 51, 57, 62, 64, 68, 70, 72, 85, 91, 96, 98, 104, 106}, {12, 16, 20, 29, 31, 37, 39, 50, 52, 58, 63, 65, 69, 71, 73, 86, 92, 97, 99, 105, 107}, {13, 17, 21, 23, 32, 34, 40, 45, 47, 53, 55, 66, 74, 79, 81, 83, 87, 89, 100, 102, 108}, {14, 18, 22, 24, 33, 35, 41, 46, 48, 54, 56, 67, 75, 80, 82, 84, 88, 90, 101, 103, 109}

S_{87} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 12, 13, 14, 23, 24, 30, 31, 38, 39, 47, 48, 55, 56, 64, 65, 72, 73, 81, 82, 89, 90, 96, 97, 106, 107, 108, 109}, {15, 16, 17, 18, 28, 29, 34, 35, 40, 41, 51, 52, 53, 54, 66, 67, 68, 69, 79, 80, 85, 86, 91, 92, 102, 103, 104, 105}, {19, 20, 21, 22, 32, 33, 36, 37, 45, 46, 49, 50, 57, 58, 62, 63, 70, 71, 74, 75, 83, 84, 87, 88, 98, 99, 100, 101}

S_{76} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 26, 43, 60, 77, 94, 111}, {10, 27, 44, 61, 78, 95, 112}, {11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109}

S_{79} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 10, 26, 27, 43, 44, 60, 61, 77, 78, 94, 95, 111, 112}, {11, 12, 13, 14, 15, 16, 28, 29, 32, 33, 34, 35, 45, 46, 47, 48, 51, 52, 55, 56, 64, 65, 66, 67, 68, 69, 74, 75, 79, 80, 83, 84, 87, 88, 96, 97, 98, 99, 100, 101, 106, 107}, {17, 18, 19, 20, 21, 22, 23, 24, 30, 31, 36, 37, 38, 39, 40, 41, 49, 50, 53, 54, 57, 58, 62, 63, 70, 71, 72, 73, 81, 82, 85, 86, 89, 90, 91, 92, 102, 103, 104, 105, 108, 109}

S_{10} := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 10, 26, 27, 43, 44, 60, 61, 77, 78, 94, 95, 111, 112}, {11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109}

S_{55} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 13, 15, 17, 19, 21, 23, 28, 30, 32, 34, 36, 38, 40, 45, 47, 49, 51, 53, 55, 57, 62, 64, 66, 68, 70, 72, 74, 79, 81, 83, 85, 87, 89, 91, 96, 98, 100, 102, 104, 106, 108}, {12, 14, 16, 18, 20, 22, 24, 29, 31, 33, 35, 37, 39, 41, 46, 48, 50, 52, 54, 56, 58, 63, 65, 67, 69, 71, 73, 75, 80, 82, 84, 86, 88, 90, 92, 97, 99, 101, 103, 105, 107, 109}

S_{74} := {1, 2, 3, 4, 5, 6, 7}, {8, 11, 15, 17, 19, 23, 27, 28, 30, 36, 40, 44, 45, 47, 51, 53, 55, 57, 61, 64, 68, 72, 76, 79, 81, 83, 89, 91, 94, 96, 100, 102, 106, 108, 110}, {9, 13, 16, 20, 21, 25, 29, 31, 32, 34, 38, 41, 43, 49, 52, 54, 56, 60, 62, 66, 70, 74, 78, 80, 82, 84, 85, 87, 92, 93, 98, 103, 104, 107, 111}, {10, 12, 14, 18, 22, 24, 26, 33, 35, 37, 39, 42, 46, 48, 50, 58, 59, 63, 65, 67, 69, 71, 73, 75, 77, 86, 88, 90, 95, 97, 99, 101, 105, 109, 112}

S_{75} := {1, 2, 3, 4, 5, 6, 7}, {8, 11, 15, 17, 19, 23, 27, 28, 30, 36, 40, 44, 45, 47, 51, 53, 55, 57, 61, 64, 68, 72, 76, 80, 81, 83, 89, 92, 94, 96, 100, 102, 106, 108, 110}, {9, 13, 16, 20, 21, 25, 29, 31, 32, 34, 38, 41, 43, 50, 52, 54, 56, 60, 62, 66, 70, 74, 78, 79, 82, 84, 85, 88, 91, 93, 98, 103, 104, 107, 111}, {10, 12, 14, 18, 22, 24, 26, 33, 35, 37, 39, 42, 46, 48, 49, 58, 59, 63, 65, 67, 69, 71, 73, 75, 77, 86, 87, 90, 95, 97, 99, 101, 105, 109, 112}

S_{59} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 13, 15, 17, 19, 21, 23, 29, 30, 32, 34, 36, 38, 41, 45, 47, 49, 51, 53, 55, 57, 63, 65, 67, 68, 70, 72, 75, 80, 82, 83, 85, 87, 90, 92, 97, 98, 100, 102, 105, 107, 109}, {12, 14, 16, 18, 20, 22, 24, 28, 31, 33, 35, 37, 39, 40, 46, 48, 50, 52, 54, 56, 58, 62, 64, 66, 69, 71, 73, 74, 79, 81, 84, 86, 88, 89, 91, 96, 99, 101, 103, 104, 106, 108}

S_{72} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 12, 13, 14, 15, 16, 28, 29, 32, 33, 34, 35, 45, 46, 47, 48, 51, 52, 55, 56, 64, 65, 66, 67, 68, 69, 74, 75, 79, 80, 83, 84, 87, 88, 96, 97, 98, 99, 100, 101, 106, 107}, {17, 18, 19, 20, 21, 22, 23, 24, 30, 31, 36, 37, 38, 39, 40, 41, 49, 50, 53, 54, 57, 58, 62, 63, 70, 71, 72, 73, 81, 82, 85, 86, 89, 90, 91, 92, 102, 103, 104, 105, 108, 109}

S_2 := {1, 2, 3, 4, 5, 6, 7}, {8, 25, 42, 59, 76, 93, 110}, {9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 111, 112}

S_3 := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 25, 26, 27, 42, 43, 44, 59, 60, 61, 76, 77, 78, 93, 94, 95, 110, 111, 112}, {11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109}

S_6 := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26, 27, 29, 31, 33, 35, 37, 39, 41, 42, 43, 44, 46, 48, 50, 52, 54, 56, 58, 59, 60, 61, 63, 65, 67, 69, 71, 73, 75, 76, 77, 78, 80, 82, 84, 86, 88, 90, 92, 93, 94, 95, 97, 99, 101, 103, 105, 107, 109, 110, 111, 112}, {11, 13, 15, 17, 19, 21, 23, 28, 30, 32, 34, 36, 38, 40, 45, 47, 49, 51, 53, 55, 57, 62, 64, 66, 68, 70, 72, 74, 79, 81, 83, 85, 87, 89, 91, 96, 98, 100, 102, 104, 106, 108}

S_7 := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26, 27, 29, 31, 33, 35, 37, 39, 41, 42, 43, 44, 46, 47, 49, 51, 54, 56, 58, 59, 60, 61, 63, 65, 66, 69, 71, 72, 75, 76, 77, 78, 80, 82, 84, 86, 87, 90, 92, 93, 94, 95, 97, 99, 101, 102, 104, 107, 109, 110, 111, 112}, {11, 13, 15, 17, 19, 21, 23, 28, 30, 32, 34, 36, 38, 40, 45, 48, 50, 52, 53, 55, 57, 62, 64, 67, 68, 70, 73, 74, 79, 81, 83, 85, 88, 89, 91, 96, 98, 100, 103, 105, 106, 108}

S_1 := {1, 2, 3, 4, 5, 6, 7}, {8 . . . , 112}

S_{113} := {1, 2, 3, 4, 5, 6, 7}, {8, 26, 44, 61, 76, 93, 111}, {9, 10, 25, 27, 42, 43, 59, 60, 77, 78, 94, 95, 110, 112}, {11, 12, 17, 18, 23, 24, 30, 31, 45, 46, 49, 50, 55, 56, 57, 58, 68, 69, 81, 82, 87, 88, 96, 97, 106, 107, 108, 109}, {13, 14, 19, 20, 34, 35, 36, 37, 38, 39, 47, 48, 62, 63, 66, 67, 72, 73, 74, 75, 83, 84, 85, 86, 100, 101, 104, 105}, {15, 21, 28, 32, 40, 51, 53, 64, 70, 79, 89, 91, 98, 102}, {16, 22, 29, 33, 41, 52, 54, 65, 71, 80, 90, 92, 99, 103}

S_{114} := {1, 2, 3, 4, 5, 6, 7}, {8, 26, 44, 61, 76, 93, 111}, {9, 10, 25, 27, 42, 43, 59, 60, 77, 78, 94, 95, 110, 112}, {11, 12, 17, 18, 23, 24, 30, 31, 45, 46, 49, 50, 55, 56, 57, 58, 68, 69, 81, 82, 87, 88, 96, 97, 106, 107, 108, 109}, {13, 14, 19, 20, 34, 35, 36, 37, 38, 39, 47, 48, 62, 63, 66, 67, 72, 73, 74, 75, 83, 84, 85, 86, 100, 101, 104, 105}, {15, 22, 28, 33, 41, 51, 54, 64, 70, 80, 90, 91, 98, 103}, {16, 21, 29, 32, 40, 52, 53, 65, 71, 79, 89, 92, 99, 102}

S_{17} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 32, 34, 36, 38, 41, 42, 43, 45, 49, 51, 53, 57, 59, 61, 63, 64, 67, 68, 70, 75, 76, 77, 79, 82, 83, 85, 87, 89, 91, 96, 100, 102, 105, 107, 108, 111, 112}, {10, 22, 26, 31, 44, 48, 56, 60, 73, 78, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 30, 33, 35, 37, 39, 40, 46, 47, 50, 52, 54, 55, 58, 62, 65, 66, 69, 71, 72, 74, 80, 81, 84, 86, 88, 90, 92, 97, 98, 101, 103, 104, 106, 109}

S_{20} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 32, 34, 36, 38, 41, 45, 47, 49, 51, 53, 57, 59, 61, 63, 64, 67, 68, 70, 72, 75, 80, 82, 83, 85, 87, 89, 92, 96, 100, 102, 105, 107, 108, 111, 112}, {10, 22, 26, 31, 42, 43, 44, 56, 60, 76, 77, 78, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 30, 33, 35, 37, 39, 40, 46, 48, 50, 52, 54, 55, 58, 62, 65, 66, 69, 71, 73, 74, 79, 81, 84, 86, 88, 90, 91, 97, 98, 101, 103, 104, 106, 109}

S_{22} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 44, 47, 49, 51, 53, 55, 59, 60, 63, 65, 66, 68, 72, 75, 76, 78, 80, 81, 83, 85, 87, 90, 92, 96, 100, 102, 104, 106, 108, 111, 112}, {10, 22, 27, 33, 43, 46, 58, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 45, 48, 50, 52, 54, 56, 57, 62, 64, 67, 69, 70, 73, 74, 79, 82, 84, 86, 88, 89, 91, 97, 98, 101, 103, 105, 107, 109}

S_{24} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 45, 48, 50, 52, 53, 55, 57, 63, 65, 66, 68, 73, 75, 76, 78, 80, 81, 83, 85, 88, 90, 92, 96, 100, 103, 104, 106, 108, 111, 112}, {10, 22, 27, 33, 42, 43, 44, 59, 60, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 47, 49, 51, 54, 56, 58, 62, 64, 67, 69, 70, 72, 74, 79, 82, 84, 86, 87, 89, 91, 97, 98, 101, 102, 105, 107, 109}

S_{25} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 45, 48, 49, 52, 53, 55, 57, 63, 65, 66, 68, 73, 75, 76, 78, 79, 81, 83, 85, 87, 90, 91, 96, 100, 103, 104, 106, 108, 111, 112}, {10, 22, 27, 33, 42, 43, 44, 59, 60, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 47, 50, 51, 54, 56, 58, 62, 64, 67, 69, 70, 72, 74, 80, 82, 84, 86, 88, 89, 92, 97, 98, 101, 102, 105, 107, 109}

S_{26} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 44, 47, 50, 51, 53, 55, 59, 60, 63, 65, 66, 68, 72, 75, 76, 78, 79, 81, 83, 85, 88, 90, 91, 96, 100, 102, 104, 106, 108, 111, 112}, {10, 22, 27, 33, 43, 46, 58, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 45, 48, 49, 52, 54, 56, 57, 62, 64, 67, 69, 70, 73, 74, 80, 82, 84, 86, 87, 89, 92, 97, 98, 101, 103, 105, 107, 109}

S_{36} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 32, 34, 36, 38, 41, 45, 47, 49, 52, 53, 57, 59, 61, 63, 64, 66, 68, 70, 72, 75, 80, 82, 83, 85, 87, 89, 92, 96, 100, 103, 104, 107, 108, 111, 112}, {10, 22, 26, 31, 42, 43, 44, 56, 60, 76, 77, 78, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 30, 33, 35, 37, 39, 40, 46, 48, 50, 51, 54, 55, 58, 62, 65, 67, 69, 71, 73, 74, 79, 81, 84, 86, 88, 90, 91, 97, 98, 101, 102, 105, 106, 109}

S_{38} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 45, 47, 49, 51, 53, 55, 57, 63, 65, 67, 68, 72, 75, 76, 78, 80, 81, 83, 85, 87, 90, 92, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 42, 43, 44, 59, 60, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 48, 50, 52, 54, 56, 58, 62, 64, 66, 69, 70, 73, 74, 79, 82, 84, 86, 88, 89, 91, 97, 98, 101, 103, 104, 107, 109}

S_{39} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 44, 48, 49, 51, 53, 55, 59, 60, 63, 65, 66, 68, 73, 75, 76, 78, 79, 81, 83, 85, 87, 90, 91, 96, 100, 102, 104, 106, 108, 111, 112}, {10, 22, 27, 33, 43, 46, 58, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 45, 47, 50, 52, 54, 56, 57, 62, 64, 67, 69, 70, 72, 74, 80, 82, 84, 86, 88, 89, 92, 97, 98, 101, 103, 105, 107, 109}

S_{40} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 45, 47, 50, 51, 53, 55, 57, 63, 65, 67, 68, 72, 75, 76, 78, 79, 81, 83, 85, 88, 90, 91, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 42, 43, 44, 59, 60, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 48, 49, 52, 54, 56, 58, 62, 64, 66, 69, 70, 73, 74, 80, 82, 84, 86, 87, 89, 92, 97, 98, 101, 103, 104, 107, 109}

S_{56} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 32, 34, 36, 38, 41, 45, 48, 49, 52, 53, 57, 59, 61, 63, 64, 66, 68, 70, 73, 75, 79, 82, 83, 85, 87, 89, 91, 96, 100, 103, 104, 107, 108, 111, 112}, {10, 22, 26, 31, 42, 43, 44, 56, 60, 76, 77, 78, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 30, 33, 35, 37, 39, 40, 46, 47, 50, 51, 54, 55, 58, 62, 65, 67, 69, 71, 72, 74, 80, 81, 84, 86, 88, 90, 92, 97, 98, 101, 102, 105, 106, 109}

S_{57} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 45, 48, 50, 51, 53, 55, 57, 63, 65, 67, 68, 73, 75, 76, 78, 80, 81, 83, 85, 88, 90, 92, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 42, 43, 44, 59, 60, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 47, 49, 52, 54, 56, 58, 62, 64, 66, 69, 70, 72, 74, 79, 82, 84, 86, 87, 89, 91, 97, 98, 101, 103, 104, 107, 109}

S_{58} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 44, 47, 49, 52, 53, 55, 59, 60, 63, 65, 67, 68, 72, 75, 76, 78, 80, 81, 83, 85, 87, 90, 92, 96, 100, 103, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 43, 46, 58, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 45, 48, 50, 51, 54, 56, 57, 62, 64, 66, 69, 70, 73, 74, 79, 82, 84, 86, 88, 89, 91, 97, 98, 101, 102, 104, 107, 109}

S_{60} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 32, 34, 36, 38, 41, 45, 48, 50, 52, 53, 57, 59, 61, 63, 64, 66, 68, 70, 73, 75, 80, 82, 83, 85, 88, 89, 92, 96, 100, 103, 104, 107, 108, 111, 112}, {10, 22, 26, 31, 42, 43, 44, 56, 60, 76, 77, 78, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 30, 33, 35, 37, 39, 40, 46, 47, 49, 51, 54, 55, 58, 62, 65, 67, 69, 71, 72, 74, 79, 81, 84, 86, 87, 90, 91, 97, 98, 101, 102, 105, 106, 109}

S_{61} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 45, 48, 49, 51, 53, 55, 57, 63, 65, 67, 68, 73, 75, 76, 78, 79, 81, 83, 85, 87, 90, 91, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 42, 43, 44, 59, 60, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 47, 50, 52, 54, 56, 58, 62, 64, 66, 69, 70, 72, 74, 80, 82, 84, 86, 88, 89, 92, 97, 98, 101, 103, 104, 107, 109}

S_{62} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 44, 47, 50, 52, 53, 55, 59, 60, 63, 65, 67, 68, 72, 75, 76, 78, 79, 81, 83, 85, 88, 90, 91, 96, 100, 103, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 43, 46, 58, 61, 71, 77, 93, 94, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 45, 48, 49, 51, 54, 56, 57, 62, 64, 66, 69, 70, 73, 74, 80, 82, 84, 86, 87, 89, 92, 97, 98, 101, 102, 104, 107, 109}

S_{63} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 43, 45, 47, 52, 53, 55, 57, 63, 65, 66, 68, 72, 75, 76, 78, 80, 81, 83, 85, 90, 92, 93, 94, 96, 100, 103, 104, 106, 108, 111, 112}, {10, 22, 27, 33, 44, 49, 59, 60, 61, 71, 77, 87, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 48, 50, 51, 54, 56, 58, 62, 64, 67, 69, 70, 73, 74, 79, 82, 84, 86, 88, 89, 91, 97, 98, 101, 102, 105, 107, 109}

S_{64} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 43, 45, 47, 51, 53, 55, 57, 63, 65, 67, 68, 72, 75, 76, 78, 80, 81, 83, 85, 90, 92, 93, 94, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 44, 49, 59, 60, 61, 71, 77, 87, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 48, 50, 52, 54, 56, 58, 62, 64, 66, 69, 70, 73, 74, 79, 82, 84, 86, 88, 89, 91, 97, 98, 101, 103, 104, 107, 109}

S_{65} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 43, 45, 47, 51, 53, 55, 57, 63, 65, 67, 68, 72, 75, 76, 78, 79, 81, 83, 85, 90, 91, 93, 94, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 44, 50, 59, 60, 61, 71, 77, 88, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 48, 49, 52, 54, 56, 58, 62, 64, 66, 69, 70, 73, 74, 80, 82, 84, 86, 87, 89, 92, 97, 98, 101, 103, 104, 107, 109}

S_{66} := {1, 2, 3, 4, 5, 6, 7}, {8, 9, 11, 13, 15, 17, 19, 23, 25, 26, 29, 30, 34, 36, 38, 41, 42, 43, 45, 48, 51, 53, 55, 57, 63, 65, 67, 68, 73, 75, 76, 78, 80, 81, 83, 85, 90, 92, 93, 94, 96, 100, 102, 105, 106, 108, 111, 112}, {10, 22, 27, 33, 44, 50, 59, 60, 61, 71, 77, 88, 95, 99, 110}, {12, 14, 16, 18, 20, 21, 24, 28, 31, 32, 35, 37, 39, 40, 46, 47, 49, 52, 54, 56, 58, 62, 64, 66, 69, 70, 72, 74, 79, 82, 84, 86, 87, 89, 91, 97, 98, 101, 103, 104, 107, 109}

A.3. Automorphism groups of non-Schurian fusions

All the non-Schurian schemes have an intransitive automorphism group. The following table contains more information:

Scheme	Rank	Group order
7	3	384
17	4	32
20	4	16
22	4	32
24	4	16
25	4	16
26	4	32
36	4	32
38	4	8
39	4	8
40	4	8
56	4	64
57	4	16
58	4	96
59	4	192
60	4	64
61	4	16
62	4	32
63	4	24
64	4	8
65	4	8
66	4	24
75	4	8
110	6	24
176	16	08

For completeness we give also the generators of these automorphism groups:

$$\begin{aligned}
\mathsf{Aut}(\#7) = \langle &(2,3)(5,10)(6,12)(7,13)(8,9)(15,16)(17,22)(19,20)(24,25)(26,27),\\
&(2,8)(3,9)(4,11)(6,7)(12,13)(15,19)(16,20)(18,21)(24,26)(25,27),\\
&(7,8)(9,13)(14,18)(19,26)(20,27)(21,28),\\
&(1,2)(5,6)(9,14)(10,15)(11,17)(12,16)(13,18)(20,21)(23,24)(27,28)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#17) = \langle &(2,3)(4,5)(8,9)(10,11)(15,20)(16,19)(17,21)(18,22)(24,26)(25,27),\\
&(2,10)(3,11)(4,8)(5,9)(6,13)(14,23)(17,25)(18,24)(21,27)(22,26),\\
&(2,8)(3,9)(4,10)(5,11)(6,13)(7,12)(17,18)(21,22)(24,25)(26,27),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#20) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,7)(3,10)(4,17)(5,25)(8,21)(9,27)(12,28)(14,16)(18,26)(19,23),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#22) = \langle &(2,3)(4,5)(8,9)(10,11)(15,20)(16,19)(17,21)(18,22)(24,26)(25,27),\\
&(2,4)(3,5)(6,13)(8,10)(9,11)(14,23)(15,20)(16,19)(17,24)(18,25)(21,26)(22,27),\\
&(2,8)(3,9)(4,10)(5,11)(6,13)(7,12)(17,18)(21,22)(24,25)(26,27),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#24) = \langle &(2,5)(3,11)(4,8)(6,13)(9,10)(14,23)(17,25)(18,27)(21,24)(22,26),\\
&(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#25) = \langle &(2,5)(3,11)(4,8)(9,10)(14,23)(15,20)(17,25)(18,27)(21,24)(22,26),\\
&(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#26) = \langle &(2,3)(4,5)(6,13)(8,9)(10,11)(16,19)(17,21)(18,22)(24,26)(25,27),\\
&(2,4)(3,5)(6,13)(8,10)(9,11)(14,23)(15,20)(16,19)(17,24)(18,25)(21,26)(22,27),\\
&(2,8)(3,9)(4,10)(5,11)(7,12)(15,20)(17,18)(21,22)(24,25)(26,27),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#36) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,2)(3,26)(4,7)(8,25)(9,14)(10,18)(11,16)(12,17)(13,15)(19,22)(23,27)(24,28),\\
&(1,4)(2,7)(5,21)(8,25)(9,19)(10,18)(11,23)(12,24)(13,15)(14,22)(16,27)(17,28)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#38) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\
&(1,23)(2,27)(3,26)(4,11)(6,20)(7,16)(8,25)(9,24)(12,19)(13,15)(14,28)(17,22)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#39) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\
&(1,23)(2,27)(3,26)(4,11)(6,20)(7,16)(8,25)(9,24)(12,19)(13,15)(14,28)(17,22)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#40) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\
&(1,23)(2,27)(3,26)(4,11)(6,20)(7,16)(8,25)(9,24)(12,19)(13,15)(14,28)(17,22)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#56) = \langle &(2,4)(3,5)(8,10)(9,11)(14,23)(16,19)(17,24)(18,25)(21,26)(22,27),\\
&(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,2)(3,26)(4,7)(8,25)(9,14)(10,18)(11,16)(12,17)(13,15)(19,22)(23,27)(24,28)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#57) = \langle &(2,5)(3,11)(4,8)(6,13)(9,10)(14,23)(17,25)(18,27)(21,24)(22,26),\\
&(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#58) = \langle &(2,8)(3,9)(4,10)(5,11)(6,13)(16,19)(17,18)(21,22)(24,25)(26,27),\\
&(2,5)(3,4)(6,13)(8,11)(9,10)(14,23)(17,26)(18,27)(21,24)(22,25),\\
&(2,3)(4,5)(7,12)(8,9)(10,11)(15,20)(17,21)(18,22)(24,26)(25,27),\\
&(1,6,28,13)(2,22,25,5)(3,24,27,8)(4,11,18,21)(7,14,12,23)(9,10,26,17)(15,16)(19,20),\\
&(1,7,15)(2,11,8)(3,26,21)(4,27,25)(6,19,14)(9,18,24)(10,17,22)(12,20,28)(13,16,23)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#59) = \langle &(2,8)(3,9)(4,10)(5,11)(15,20)(16,19)(17,18)(21,22)(24,25)(26,27),\\
&(2,4)(3,5)(8,10)(9,11)(14,23)(16,19)(17,24)(18,25)(21,26)(22,27),\\
&(2,3)(4,5)(6,13)(7,12)(8,9)(10,11)(17,21)(18,22)(24,26)(25,27),\\
&(1,2,18,28,24,10)(3,16,11,26,19,22)(4,25,12,17,8,7)(5,14,27,21,23,9)(13,15,20)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#60) = \langle &(2,4)(3,5)(8,10)(9,11)(14,23)(16,19)(17,24)(18,25)(21,26)(22,27),\\
&(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,2)(3,26)(4,7)(8,25)(9,14)(10,18)(11,16)(12,17)(13,15)(19,22)(23,27)(24,28)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#61) = \langle &(2,5)(3,11)(4,8)(9,10)(14,23)(15,20)(17,25)(18,27)(21,24)(22,26),\\
&(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathsf{Aut}(\#62) = \langle &(2,8)(3,9)(4,10)(5,11)(15,20)(16,19)(17,18)(21,22)(24,25)(26,27),\\
&(2,5)(3,4)(8,11)(9,10)(14,23)(15,20)(17,26)(18,27)(21,24)(22,25),\\
&(2,3)(4,5)(6,13)(7,12)(8,9)(10,11)(17,21)(18,22)(24,26)(25,27),\\
&(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26)\rangle
\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#63) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,2,26,23,24,8)(3,14,9,25,28,27)(4,17,11)(5,15,19,10,6,7)(12,18,13,16,21,20)\rangle\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#64) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\ &(1,23)(2,27)(3,26)(4,11)(6,20)(7,16)(8,25)(9,24)(12,19)(13,15)(14,28)(17,22)\rangle\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#65) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\ &(1,23)(2,27)(3,26)(4,11)(6,20)(7,16)(8,25)(9,24)(12,19)(13,15)(14,28)(17,22)\rangle\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#66) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,2,25,23,24,3)(5,15,19,10,6,7)(8,14,9,26,28,27)(11,17,22)(12,18,13,16,21,20)\rangle\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#75) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\ &(1,23)(2,27)(3,26)(4,11)(6,20)(7,16)(8,25)(9,24)(12,19)(13,15)(14,28)(17,22)\rangle\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#110) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,2,8)(3,14,9)(4,16,13)(6,17,12)(7,15,11)(10,18,21)(19,20,22)(23,27,25)(24,26,28),\\ &(1,23)(2,24)(3,25)(5,10)(6,15)(7,19)(8,26)(9,27)(12,16)(13,20)(14,28)(18,21)\rangle\end{aligned}$$

$$\begin{aligned}\mathsf{Aut}(\#176) = \langle &(2,9)(3,8)(4,11)(5,10)(6,13)(7,12)(15,20)(16,19)(17,22)(18,21)(24,27)(25,26),\\ &(1,14)(3,8)(4,17)(5,18)(6,15)(7,16)(10,21)(11,22)(12,19)(13,20)(23,28)(25,26),\\ &(1,23)(2,24)(3,25)(5,10)(6,15)(7,19)(8,26)(9,27)(12,16)(13,20)(14,28)(18,21)\rangle\end{aligned}$$

A.4. *Automorphism groups of Schurian fusions*

For the Schurian schemes we compiled the following table. Using the indicated id, the automorphism groups can quickly be obtained in gap4 using `TransitiveGroup(x,y)`.

Scheme	Rank	Group order	Id of group in transitive-groups-library of gap4
2	3	$2^{25} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	[28, 1834]
3	3	$2^{25} \cdot 3^9 \cdot 5 \cdot 7$	[28, 1798]
6	3	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[28, 502]
10	4	$2^{25} \cdot 3^2 \cdot 5 \cdot 7$	[28, 1736]
55	4	$2^6 \cdot 3 \cdot 7$	[28, 159]
72	4	$2^{21} \cdot 3^8 \cdot 7$	[28, 1757]
74	4	$2^3 \cdot 3^2 \cdot 7$	[28, 70]
76	5	$2^{18} \cdot 3^2 \cdot 5 \cdot 7$	[28, 1659]
79	5	$2^{21} \cdot 3 \cdot 7$	[28, 1619]
87	5	$2^{22} \cdot 3^7 \cdot 7$	[28, 1753]
91	6	$2^{14} \cdot 3 \cdot 7$	[28, 1005]
95	6	$2^{22} \cdot 7$	[28, 1583]
109	6	$2^3 \cdot 3 \cdot 7$	[28, 27]
111	7	$2^{15} \cdot 7$	[28, 914]
113	7	$2^8 \cdot 7$	[28, 185]
114	7	$2^8 \cdot 7$	[28, 183]
138	8	$2^{21} \cdot 3^7 \cdot 7$	[28, 1743]
145	9	$2^{21} \cdot 7$	[28, 1520]
149	10	$2^{14} \cdot 7$	[28, 746]
155	12	$2^7 \cdot 7$	[28, 110]
175	16	$2^3 \cdot 7$	[28, 11]

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